

A NEW NUMERICAL TECHNIQUE FOR SOLVING FRACTIONAL BRATU'S INITIAL VALUE PROBLEMS IN THE CAPUTO AND CAPUTO-FABRIZIO SENSE

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Abstract. The purpose of this paper is to propose a new numerical technique called the natural decomposition method (NDM) for solving fractional Bratu's initial value problems (FBIVP) in the Caputo and Caputo-Fabrizio sense. The NDM is a combined form of the natural transform method and the Adomian decomposition method. The numerical example is provided in order to validate the efficiency and reliability of the proposed method. The obtained results reveal that the proposed method is a very efficient and simple tool for solving fractional differential equations.

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1. Introduction

It is well known that the classical Bratu-type equation is of the form

$$u''(t) + \lambda \exp(u(t)) = 0, 0 < t < 1. \quad (1)$$

The exact solution is [1]

$$u(t) = -2 \ln \left[\frac{\cosh \left(\left(t - \frac{1}{2} \right) \frac{\theta}{2} \right)}{\cosh \left(\frac{\theta}{4} \right)} \right], \quad (2)$$

where θ satisfies

$$\theta = \sqrt{2\lambda} \cosh \left(\frac{\theta}{4} \right). \quad (3)$$

The problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$, and $\lambda < \lambda_c$, respectively, where the critical value λ_c satisfies the equation

$$1 = \frac{1}{4} \sqrt{2\lambda_c} \sinh\left(\frac{\theta_c}{4}\right), \quad (4)$$

where

$$\lambda_c = 3.513830719. \quad (5)$$

The classical Bratu-type equation was used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of the thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [1–4].

Several authors have proposed various methods to obtain the approximate solution of the classical Bratu-type equation such as: the Adomian decomposition method (ADM) [4], variational iteration method (VIM) [5], homotopy perturbation method (HPM) [6], optimal homotopy analysis method (OHAM) [7], Optimal perturbation iteration method (OPIM) [8], Lie-group shooting method (LGSM) [9].

The main objective of this paper is to propose a powerful method called the natural decomposition method (NDM) for solving the fractional Bratu's initial value problems (FBIVP) with two different fractional derivative operators in the forms

$$\begin{cases} D^\alpha u(t) + \lambda \exp(u(t)) = 0, \\ u(0) = a_0, u'(0) = a_1, \end{cases} \quad (6)$$

where D^α is the Caputo fractional derivative operator of order α , $1 < \alpha \leq 2$, $0 < t < 1$ and $\lambda \in \mathbb{R}$, and

$$\begin{cases} \mathcal{D}^{(\mu)} u(t) + \lambda \exp(u(t)) = 0, \\ u(0) = a_0, u'(0) = a_1, \end{cases} \quad (7)$$

where $\mathcal{D}^{(\mu)} = \mathcal{D}^{(\alpha+1)}$ is the Caputo-Fabrizio fractional derivative operator of order $\mu = \alpha + 1$, $0 < \alpha \leq 1$.

In recent years, many numerical and approximate analytical methods have been developed for solving fractional differential equations in the Caputo and Caputo-Fabrizio sense. For more details, see [10–18].

2. Preliminaries and notations

In this section, we present the main definitions and properties of the fractional calculus theory and natural transform.

Definition 1 [19] Let $u \in L^1(0, 1)$. The left sided Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined by

$$I^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau) d\tau, & \alpha > 0, \\ u(t), & \alpha = 0, \end{cases} \quad (8)$$

where $\Gamma(\cdot)$ is the Euler gamma function. □

Definition 2 [19] Let $u^{(n)} \in L^1(0, 1)$. The left sided Caputo fractional derivative of order $\alpha \geq 0$ is defined by

$$D^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau, & n - 1 < \alpha < n, \\ u^{(n)}(t), & \alpha = n, \end{cases} \quad (9)$$

where $n = [\alpha] + 1$ with $[\alpha]$ being the integer part of α . □

In equation (9) if transformations happen as follows

$$(t - \tau)^{n-\alpha-1} \longrightarrow \exp\left[-\frac{\alpha(t - \tau)}{1 - \alpha}\right] \text{ and } \frac{1}{\Gamma(n - \alpha)} \longrightarrow \frac{M(\alpha)}{1 - \alpha}, \quad (10)$$

the new definition of the fractional operator is expressed by Caputo and Fabrizio.

Definition 3 [20] Let $u \in H^1(0, 1)$, then the Caputo-Fabrizio fractional derivative of order $\alpha, 0 < \alpha \leq 1$ is defined as

$$\mathcal{D}^{(\alpha)} u(t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t u'(\tau) \exp\left[-\frac{\alpha(t - \tau)}{1 - \alpha}\right] d\tau, \quad (11)$$

where $M(\alpha)$ is a normalization function that satisfies $M(0) = M(1) = 1$. □

From equation (11) it follows that if $u(t) = C$ is a constant, then $\mathcal{D}^{(\alpha)} C = 0$ as in the sense of Caputo [19].

If $u \notin H^1(0, 1)$, then its fractional derivative is redefined as [20]

$$\mathcal{D}^{(\alpha)} u(t) = \frac{\alpha M(\alpha)}{1 - \alpha} \int_0^t (u(t) - u(\tau)) \exp\left[-\frac{\alpha(t - \tau)}{1 - \alpha}\right] d\tau, \quad 0 < t < 1. \quad (12)$$

For $n \geq 1$ and $0 < \alpha \leq 1$, the fractional derivative of order $(\alpha + n)$ is defined by

$$\mathcal{D}^{(\alpha+n)} u(t) = \mathcal{D}^{(\alpha)} (\mathcal{D}^{(n)} u(t)). \quad (13)$$

The above Caputo-Fabrizio fractional derivative was later modified by Jorge Losada and Juan José Nieto [21] as

$$\mathcal{D}^{(\alpha)}u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t u'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, 0 < t < 1. \quad (14)$$

The fractional integral corresponding to the derivative in equation (14) was defined by Jorge Losada and Juan José Nieto in 2015, as follows.

Definition 4 [21] Let $0 < \alpha \leq 1$. The fractional integral of order α of u is defined by

$$\mathcal{I}^{(\alpha)}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(\tau) d\tau, 0 < t < 1. \quad (15)$$

From the definition in equation (15), the fractional integral of the Caputo-Fabrizio type of a function u of order $0 < \alpha \leq 1$ is an average between function u and its one order integral, i.e.,

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1. \quad (16)$$

Therefore,

$$M(\alpha) = \frac{2}{2-\alpha}, 0 < \alpha \leq 1. \quad (17)$$

Due to this, Losada and Nieto remarked that the Caputo-Fabrizio fractional derivative can be redefined as

Definition 5 [21] Let $0 < \alpha \leq 1$. The fractional Caputo-Fabrizio derivative of order α of a function u is given by

$$\mathcal{D}^{(\alpha)}u(t) = \frac{1}{1-\alpha} \int_0^t u'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, 0 < t < 1. \quad (18)$$

Definition 6 [22] The natural transform is defined over the set of functions

$$A = \left\{ u(t) / \exists M, \xi_1, \xi_2 > 0, |u(t)| < M \exp\left(\frac{|t|}{\xi_j}\right), \text{ if } t \in (-1)^j \times [0, \infty) \right\}, \quad (19)$$

by the following integral

$$\mathcal{N}^+[u(t)] = R^+(s, v) = \frac{1}{v} \int_0^{+\infty} \exp\left(-\frac{st}{v}\right) u(t) dt, s, v \in (0, \infty). \quad (20)$$

Some basic properties of the natural transform are given as follows

Property 1: The natural transform is a linear operator

$$\mathcal{N}^+ [au(t) \pm bw(t)] = a\mathcal{N}^+ [u(t)] \pm b\mathcal{N}^+ [w(t)], a, b \in \mathbb{R}. \quad (21)$$

Property 2: If $u^{(n)}(t)$ is the n -th derivative of function $u(t)$ with respect to the time variable t , then its natural transform is given by

$$\mathcal{N}^+ [u^{(n)}(t)] = R_n^+(s, v) = \frac{s^n}{v^n} R^+(s, v) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{v^{n-k}} u^{(k)}(0). \quad (22)$$

Property 3: (Convolution property) Suppose $U^+(s, v)$ and $W^+(s, v)$ are the natural transforms of $u(t)$ and $w(t)$, respectively, both defined in the set A . Then the natural transform of their convolution is given by

$$\mathcal{N}^+ [(u * w)(t)] = vU^+(s, v)W^+(s, v). \quad (23)$$

Property 4: Some special natural transforms

$$\mathcal{N}^+ [1] = \frac{1}{s}, \mathcal{N}^+ [t] = \frac{v}{s^2}, \mathcal{N}^+ \left[\frac{t^n}{n!} \right] = \frac{v^n}{s^{n+1}}, n = 0, 1, 2, \dots \quad (24)$$

Property 5: If $\alpha > -1$, then the natural transform of t^α is given by

$$\mathcal{N}^+ [t^\alpha] = \Gamma(\alpha + 1) \frac{v^\alpha}{s^{\alpha+1}}. \quad (25)$$

Theorem 1 [23] Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$ and $R^+(s, v)$ be the natural transform of the function $u(t)$, then the natural transform denoted by $R_\alpha^+(s, v)$ of the Caputo fractional derivative of the function $u(t)$ of order α , is given by

$$\mathcal{N}^+ [D^\alpha u(t)] = R_\alpha^+(s, v) = \frac{s^\alpha}{v^\alpha} R^+(s, v) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{v^{\alpha-k}} u^{(k)}(0). \quad (26)$$

Theorem 2 The natural transform of the Caputo-Fabrizio fractional derivative of the function $u(t)$ of order $\alpha + n$, where $0 < \alpha \leq 1$ and $n \in \mathbb{N} \cup \{0\}$, is given by

$$\mathcal{N}^+ \left[\mathcal{D}^{(\alpha+n)} u(t) \right] = \frac{1}{s - \alpha(s - v)} \left[\frac{s^{n+1}}{v^n} \mathcal{N}^+ (u(t)) - \sum_{k=0}^n \left(\frac{s}{v} \right)^{n-k} u^{(k)}(0) \right]. \quad (27)$$

PROOF By the Definition of the Caputo-Fabrizio fractional derivative 5 and using the relation (13), we have

$$\begin{aligned}
\mathcal{N}^+ \left[\mathcal{D}^{(\alpha+n)} u(t) \right] &= \mathcal{N}^+ \left[\mathcal{D}^{(\alpha)} (\mathcal{D}^{(n)} u(t)) \right] \\
&= \frac{1}{1-\alpha} \frac{1}{v} \int_0^{+\infty} \exp\left(-\frac{st}{v}\right) \int_0^t u^{(n)}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\
&= \frac{1}{1-\alpha} \frac{1}{v} \int_0^{+\infty} \exp\left(-\frac{st}{v}\right) \left(u^{(n)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right] \right) \\
&= \frac{1}{1-\alpha} \mathcal{N}^+ \left(u^{(n)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right] \right).
\end{aligned}$$

Hence, from the Properties 2 and 3 of the natural transform, we have

$$\begin{aligned}
\mathcal{N}^+ \left[\mathcal{D}^{(\alpha+n)} u(t) \right] &= \frac{1}{1-\alpha} v \mathcal{N}^+ \left(u^{(n)}(t) \right) \mathcal{N}^+ \left(\exp\left[-\frac{\alpha t}{1-\alpha}\right] \right) \\
&= \frac{v}{s-\alpha(s-v)} \left[\frac{s^n}{v^n} \mathcal{N}^+ (u(t)) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{v^{n-k}} u^{(k)}(0) \right] \\
&= \frac{v}{s-\alpha(s-v)} \left[\frac{s^{n+1}}{v^{n+1}} \mathcal{N}^+ (u(t)) - \sum_{k=0}^n \frac{s^{n-k}}{v^{n-k}} u^{(k)}(0) \right] \\
&= \frac{1}{s-\alpha(s-v)} \left[\frac{s^{n+1}}{v^n} \mathcal{N}^+ (u(t)) - \sum_{k=0}^n \left(\frac{s}{v}\right)^{n-k} u^{(k)}(0) \right]. \blacksquare
\end{aligned}$$

3. Description of the natural decomposition method (NDM)

Theorem 3 Consider the following FBIVP (6) and (7). Then, by NDM the solution of (6) and (7) is given in the form of infinite series as follows

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \quad (28)$$

PROOF Consider the following FBIVP (6) and (7). First, we define the nonlinear operator

$$Ru(t) = \exp(u(t)). \quad (29)$$

1) For the Caputo fractional derivative

Equation (6) is written in the form

$$D^\alpha u(t) + \lambda Ru(t) = 0, 0 < t < 1, \lambda \in \mathbb{R}. \quad (30)$$

Applying the natural transform on both sides of (30) and using the Theorem 1, we get

$$\mathcal{N}^+[u(t)] = \frac{1}{s}u(0) + \frac{v}{s^2}u'(0) - \frac{v^\alpha}{s^\alpha}\mathcal{N}^+[\lambda Ru(t)]. \quad (31)$$

After that, let us take the inverse natural transform on both sides of (31), we have

$$u(t) = a_0 + a_1t - \lambda \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+[Ru(t)]\right). \quad (32)$$

Now, we represent the solution in an infinite series form

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \quad (33)$$

and the nonlinear terms $Ru(t)$ can be decomposed as

$$Ru(t) = \sum_{n=0}^{\infty} A_n(t), \quad (34)$$

where A_n are the Adomian polynomials [24] of $u_0, u_1, u_2, \dots, u_n$, and it can be calculated by formula given below

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[R \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (35)$$

Using equations (33) and (34), we can rewrite equation (32) as

$$\sum_{n=0}^{\infty} u_n(t) = a_0 + a_1t - \lambda \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+\left[\sum_{n=0}^{\infty} A_n(t)\right]\right). \quad (36)$$

By comparing both sides of equation (35), we have the following relation

$$\begin{aligned} u_0(t) &= a_0 + a_1t, \\ u_1(t) &= -\lambda \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+[A_0(t)]\right), \\ u_2(t) &= -\lambda \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+[A_1(t)]\right), \\ &\vdots \end{aligned}$$

In general, the recursive relation is given by

$$\begin{aligned} u_0(t) &= a_0 + a_1t, \\ u_{n+1}(t) &= -\lambda \mathcal{N}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathcal{N}^+[A_n(t)]\right), \quad n \geq 0. \end{aligned} \quad (37)$$

Then, the solution of FBIVP (6) is given in the form of infinite series as follows

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \quad (38)$$

2) For the Caputo-Fabrizio fractional derivative

Equation (7) is written in the form

$$\mathcal{D}^{(\alpha+1)}u(t) + \lambda Ru(t) = 0, 0 < t < 1, \lambda \in \mathbb{R}. \quad (39)$$

Applying the natural transform on both sides of (39) and using the Theorem 2, we get

$$\mathcal{N}^+[u(t)] = \frac{1}{s}u(0) + \frac{v}{s^2}u'(0) - \frac{v(s - \alpha(s - v))}{s^2}\mathcal{N}^+[\lambda Ru(t)]. \quad (40)$$

After that, let us take the inverse natural transform on both sides of (40), we have

$$u(t) = a_0 + a_1t - \lambda \mathcal{N}^{-1}\left(\frac{v(s - \alpha(s - v))}{s^2}\mathcal{N}^+[Ru(t)]\right). \quad (41)$$

Now, we represent the solution in an infinite series form

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \quad (42)$$

and the nonlinear terms $Ru(t)$ can be decomposed as

$$Ru(t) = \sum_{n=0}^{\infty} A_n(t). \quad (43)$$

Using equations (42) and (43), we can rewrite equation (41) as

$$\sum_{n=0}^{\infty} u_n(t) = a_0 + a_1t - \lambda \mathcal{N}^{-1}\left(\frac{v(s - \alpha(s - v))}{s^2}\mathcal{N}^+\left[\sum_{n=0}^{\infty} A_n(t)\right]\right). \quad (44)$$

By comparing both sides of equation (44), we have the following relation

$$\begin{aligned} u_0(t) &= a_0 + a_1t, \\ u_1(t) &= -\lambda \mathcal{N}^{-1}\left(\frac{v(s - \alpha(s - v))}{s^2}\mathcal{N}^+[A_0(t)]\right), \\ u_2(t) &= -\lambda \mathcal{N}^{-1}\left(\frac{v(s - \alpha(s - v))}{s^2}\mathcal{N}^+[A_1(t)]\right), \\ &\vdots \end{aligned}$$

In general the recursive relation is given by

$$\begin{aligned} u_0(t) &= a_0 + a_1 t, \\ u_{n+1}(t) &= -\lambda \mathcal{N}^{-1} \left(\frac{v(s - \alpha(s - v))}{s^2} \mathcal{N}^+ [A_n(t)] \right), \quad n \geq 0. \end{aligned} \quad (45)$$

Then, the solution of FBIVP (7) is given in the form of infinite series as follows

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \quad (46)$$

Theorem 4 *Let \mathcal{B} be a Banach space. Then the series solution of the FBIVP (6) and (7) converges to $S \in \mathcal{B}$, if there exists γ , $0 < \gamma < 1$ such that*

$$\|u_n\| \leq \gamma \|u_{n-1}\|, \quad \forall n \in \mathbb{N}. \quad (47)$$

PROOF Define the sequences $\{S_n\}_{n \geq 1}$ of partial sums of the series given by the recursive relation (37) or (45) as

$$S_n(t) = u_1(t) + u_2(t) + u_3(t) + \dots + u_n(t), \quad (48)$$

and we need to show that $\{S_n\}_{n \geq 1}$ are a Cauchy sequences in Banach space \mathcal{B} . For this purpose, we consider

$$\|S_{n+1} - S_n\| \leq \|u_{n+1}\| \leq \gamma \|u_n\| \leq \gamma^2 \|u_{n-1}\| \leq \dots \leq \gamma^{n+1} \|u_0\|. \quad (49)$$

For every $n, m \in \mathbb{N}$, $n \geq m$, by using (47) and triangle inequality successively, we have

$$\begin{aligned} \|S_n - S_m\| &= \|S_{m+1} - S_m + S_{m+2} - S_{m+1} + \dots + S_n - S_{n-1}\| \\ &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \dots + \|S_n - S_{n-1}\| \\ &\leq \gamma^{m+1} \|u_0\| + \gamma^{m+2} \|u_0\| + \dots + \gamma^n \|u_0\| \\ &= \gamma^{m+1} (1 + \gamma + \dots + \gamma^{n-m-1}) \|u_0\| \leq \gamma^{m+1} \left(\frac{1 - \gamma^{n-m}}{1 - \gamma} \right) \|u_0\|. \end{aligned}$$

Since $0 < \gamma < 1$, so $1 - \gamma^{n-m} \leq 1$ then

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1 - \gamma} \|u_0\|. \quad (50)$$

Since u_0 is bounded, then

$$\lim_{n, m \rightarrow \infty} \|S_n - S_m\| = 0. \quad (51)$$

Therefore, the sequences $\{S_n\}_{n \geq 1}$ are Cauchy sequences in Banach space \mathcal{B} , so the series solution defined in (28) converges. ■

Example 1. Consider the following fractional Bratu's initial value problem with Caputo fractional derivative

$$\begin{cases} D^\alpha u(t) - 2\exp(u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, \quad 1 < \alpha \leq 2. \end{cases} \quad (52)$$

For $\alpha = 2$, the exact solution of BIVP (52), is (see [25])

$$u(t) = -2\ln(\cos t).$$

Following the description of the NDM presented in Section 3, gives

$$\sum_{n=0}^{\infty} u_n(t) = 2\mathcal{N}^{-1} \left(\frac{v^\alpha}{s^\alpha} \mathcal{N}^+ \left[\sum_{n=0}^{\infty} A_n(t) \right] \right), \quad (53)$$

where $\exp(u(t)) = \sum_{n=0}^{\infty} A_n(t)$ are the Adomian polynomials that represents the nonlinear term, and it can be calculated by formula (35). For the convenience of the reader, the first few nonlinear terms are as follows:

$$\begin{aligned} A_0(t) &= \exp(u_0(t)), \\ A_1(t) &= u_1(t) \exp(u_0(t)), \\ A_2(t) &= \left(u_2(t) + \frac{1}{2}u_1^2(t) \right) \exp(u_0(t)), \\ A_3(t) &= \left(u_3(t) + u_1(t)u_2(t) + \frac{1}{6}u_1^3(t) \right) \exp(u_0(t)), \\ &\vdots \end{aligned}$$

From the relationship in (37) and Property 5, we have

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= \frac{2}{\Gamma(\alpha+1)} t^\alpha, \\ u_2(t) &= \frac{4}{\Gamma(2\alpha+1)} t^{2\alpha}, \\ u_3(t) &= \frac{4 [2\Gamma^2(\alpha+1) + \Gamma(2\alpha+1)]}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} t^{3\alpha}, \\ &\vdots \end{aligned}$$

Therefore, the approximate solution of FBIVP (52), is given by

$$u(t) = \frac{2}{\Gamma(\alpha+1)} t^\alpha + \frac{4}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{4 [2\Gamma^2(\alpha+1) + \Gamma(2\alpha+1)]}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} t^{3\alpha} + \dots \quad (54)$$

Now, we consider the following fractional Bratu's initial value problem with Caputo-Fabrizio fractional derivative

$$\begin{cases} \mathcal{D}^{(\mu)}u(t) - 2\exp(u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, \mu = \alpha + 1, 0 < \alpha \leq 1. \end{cases} \quad (55)$$

For $\mu = 2$ ($\alpha = 1$), the exact solution of BIVP (55), is (see [25])

$$u(t) = -2\ln(\cos t).$$

Following the description of the NDM presented in Section 3, gives

$$\sum_{n=0}^{\infty} u_n(t) = 2 \mathcal{N}^{-1} \left(\frac{v(s - \alpha(s - v))}{s^2} \mathcal{N}^+ \left[\sum_{n=0}^{\infty} A_n(t) \right] \right). \quad (56)$$

Similarly above, and from the relationship in (45) and Property 4, we have

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= 2(1 - \alpha)t + \alpha t^2, \\ u_2(t) &= 2(1 - \alpha)^2 t^2 + \frac{4\alpha(1 - \alpha)}{3} t^3 + \frac{\alpha^2}{6} t^4, \\ u_3(t) &= \frac{8}{3}(1 - \alpha)^3 t^3 + \frac{7}{3}\alpha(1 - \alpha)^2 t^4 + \frac{3}{5}\alpha^2(1 - \alpha)t^5 + \frac{2}{45}\alpha^3 t^6, \\ &\vdots \end{aligned}$$

Therefore, the approximate solution of FBIVP (55), is given by

$$u(t) = \sum_{n=0}^{\infty} u_n(t) = u_0(t) + u_1(t) + u_2(t) + u_3(t) + \dots \quad (57)$$

Table 1. Numerical comparison between the 4-term approximate solutions by NDM Caputo at $\alpha = 2$, NDM Caputo-Fabrizio at $\alpha = 1$ and exact solution

t	u_{NDM-C}	u_{NDM-CF}	Exact solution	$ u_{exact} - u_{NDM-C} $	$ u_{exact} - u_{NDM-CF} $
0.1	0.010017	0.010017	0.010017	1.3536×10^{-10}	1.3536×10^{-10}
0.2	0.040270	0.040270	0.040270	3.4994×10^{-8}	3.4994×10^{-8}
0.3	0.091382	0.091382	0.091383	9.1185×10^{-7}	9.1185×10^{-7}
0.4	0.16445	0.16445	0.16446	9.3270×10^{-6}	9.3270×10^{-6}
0.5	0.26111	0.26111	0.26117	5.7370×10^{-5}	5.7370×10^{-5}

Figure 1, represents the behavior of the 4-term approximate solutions by NDM Caputo, NDM Caputo-Fabrizio and exact solution to some different values of α . Figure 2, represents the comparison of the exact solution and 4-term approximate solutions by NDM Caputo at $\alpha = 2$ and NDM Caputo-Fabrizio at $\alpha = 1$. Table 1, represents the numerical comparison between the 4-term approximate solutions by

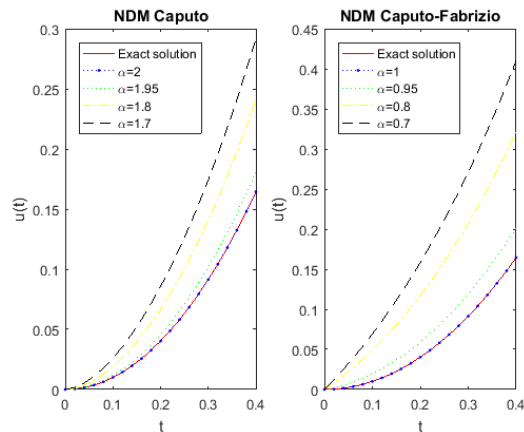


Fig. 1. The behavior of the exact solution and 4-term approximate solutions by NDM Caputo and NDM Caputo-Fabrizio for different values of α

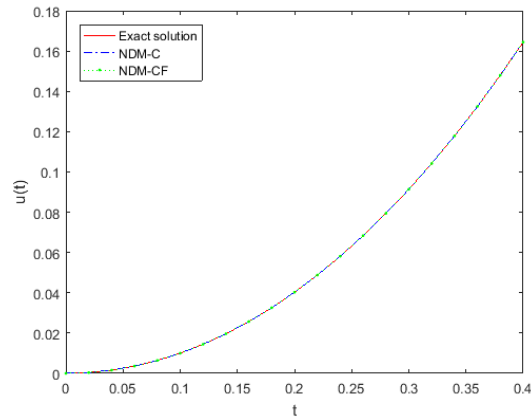


Fig. 2. Comparison of the exact solution and 4-term approximate solutions by NDM Caputo at $\alpha = 2$ and NDM Caputo-Fabrizio at $\alpha = 1$

NDM Caputo at $\alpha = 2$ and NDM Caputo-Fabrizio at $\alpha = 1$ and exact solution for different values of t . Consequently the approximate solutions are rapidly convergent as the exact solution. In this paper, we only apply four terms to approximate the solutions, if we apply more terms of the approximate solutions, the accuracy of the approximate solutions will be greatly improved.

4. Conclusions

In this paper, the natural decomposition method (NDM) is used to solve the fractional Bratu's initial value problems in the Caputo and Caputo-Fabrizio sense. In order to illustrate the accuracy and effectiveness of the proposed method, it is

applied to a numerical example. The approximate solutions obtained by this method are compared with the exact solutions. The numerical results obtained are in an excellent agreement with the exact solutions, therefore, we can conclude that the NDM is practically well suited for solving a wide range of fractional differential equations.

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