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RECURRENCE RELATIONS FOR A MULTI-CHANNEL CLOSED QUEUEING SYSTEM WITH ERLANGIAN SERVICE TIMES OF SECOND ORDER

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Abstract. We propose a method for determining the steady-state characteristics of a multi-channel closed queueing system with exponential distribution of the time generation of service requests and the second order Erlang distributions of the service times. Recurrence relations to compute the steady-state distribution of the number of customers are obtained. The developed algorithms are tested on examples using simulation models constructed with the assistance of the GPSS World tools.

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1. Introduction

Closed queueing systems are widely used as models to evaluate characteristics of information systems, data networks and queueing processes in production, transport, trade, logistics and service systems [1]. The closed system is also called the system with a finite number of sources or an Engset system.

Suppose that a queueing system with n channels receives service requests from m identical sources. Each source is alternately on and off. A source is off when it has a service request being served, otherwise the source is on. A source in the on-state generates a new service request after an exponentially distributed time (the generation time) with mean $1/\lambda$. The sources act independently of each other. The service time of a service request has the second order Erlang distribution. A service request, that is generated when n channels are occupied, waits in the queue.

To investigate the systems with Erlangian service times, in particular the $M/E_s/1/\infty$ system [2], the method of fictitious phases, developed by A.K. Erlang

[3], was applied. The Erlangian service times of the second order means that each customer runs two service phases sequentially, the duration of which is distributed exponentially with parameters μ_1 and μ_2 respectively.

The objective of this work is the construction with the aid of a fictitious phase method that is a recursive algorithm for computing the steady-state distribution of the number of customers in the multi-channel closed queueing system. We consider the system with an exponential distribution of the time generation of service requests and Erlangian service times of the second order. A similar approach is used in [4, 5], where recursive algorithms are developed for the systems $M/E_2/2/m$, $M/E_2/2/\infty$, $M/E_2/3/m$ and $M/E_2/3/\infty$ as well as for the systems of the same types with threshold and hysteretic strategies of the random dropping of customers.

2. Construction of a recursive algorithm

Suppose that the service time of each customer is distributed under the generalized Erlang law of the second order, that is, the service time is the sum of two independent random variables exponentially distributed with parameters μ_1 and μ_2 respectively.

Let n_c denote the number of customers in the system. Based on the phase method, we introduce the following designations for system states: s_0 signifies that customers are absent in the system; $s_{k(ij)}$ signifies that there are k customers in the system $(1 \le k \le m)$ and that i customers are at the first phase of service and j customers are at the second phase $(0 \le i \le n_c \le n,\ 0 \le j \le n_c \le n,\ 1 \le i + j \le n_c \le n$ or $0 \le i \le n \le n_c,\ 0 \le j \le n \le n_c,\ 1 \le i + j \le n \le n_c)$. We denote steady-state probabilities that the system is in the states s_0 and $s_{k(ij)}$ by p_0 and $p_{k(ij)}$, respectively. Then we obtain the following system of equations for determining these probabilities:

$$
\begin{aligned}
&-\lambda p_0 + \mu_2 p_{1(01)} = 0;\\
&-(\lambda_k + k\mu_2)p_{k(0k)} + \mu_1 p_{k(1,k-1)} + (k+1)\mu_2 p_{k+1(0,k+1)} = 0, \quad 1 \le k \le n-1;\\
&-(\lambda_k + k\mu_1)p_{k(k0)} + \lambda_{k-1} p_{k-1(k-1,0)} + \mu_2 p_{k+1(k1)} = 0, \quad 1 \le k \le n-1; \qquad p_{0(00)} = p_0;\\
&-\left(\lambda_k + (k-j)\mu_1 + j\mu_2\right) p_{k(k-j,j)} + \lambda_{k-1} p_{k-1(k-j-1,j)} + (k-j+1)\mu_1 p_{k(k-j+1,j-1)} +\\
&\qquad\qquad + (j+1)\mu_2 p_{k+1(k-j,j+1)} = 0, \quad 1 \le j \le k-1, \quad j+1 \le k \le n-1;\\
&-(\lambda_n + n\mu_1)p_{n(n0)} + \lambda_{n-1} p_{n-1(n-1,0)} + \mu_2 p_{n+1(n-1,1)} = 0;\\
&-(\lambda_k + n\mu_1)p_{k(n0)} + \lambda_{k-1} p_{k-1(n0)} + \mu_2 p_{k+1(n-1,1)} = 0, \quad n+1 \le k \le m-1;\\
&-n\mu_1 p_{m(n0)} + \lambda p_{m-1(n0)} = 0;\\
&-\left(\lambda_n + (n-j)\mu_1 + j\mu_2\right) p_{n(n-j,j)} + \lambda_{n-1} p_{n-1(n-j-1,j)} + (n-j+1)\mu_1 p_{n(n-j+1,j-1)} +\\
&\qquad\qquad + (j+1)\mu_2 p_{n+1(n-j-1,j+1)} = 0, \quad 1 \le j \le n-1;
\end{aligned}
\tag{1}
$$

$$-\left(\lambda_k+(n-j)\mu_1+j\mu_2\right)p_{k(n-j,j)}+\lambda_{k-1}p_{k-1(n-j,j)}+(n-j+1)\mu_1 p_{k(n-j+1,j-1)}+$$
$$+(j+1)\mu_2 p_{k+1(n-j-1,j+1)}=0,\quad n+1\le k\le m-1,\quad 1\le j\le n-1;$$

$$\begin{aligned}
&-(\lambda_n+n\mu_2)p_{n(0n)}+\mu_1 p_{n(1,n-1)}=0;\\
&-(\lambda_k+n\mu_2)p_{k(0n)}+\lambda_{k-1}p_{k-1(0n)}+\mu_1 p_{k(1,n-1)}=0,\quad n+1\le k\le m-1;\\
&-n\mu_2 p_{n+m(0n)}+\lambda p_{m-1(0n)}+\mu_1 p_{m(1,n-1)}=0;\\
&-\left((n-j)\mu_1+j\mu_2\right)p_{n+m(n-j,j)}+\lambda p_{m-1(n-j,j)}+\\
&\qquad+(n-j+1)\mu_1 p_{m(n-j+1,j-1)}=0,\quad 1\le j\le n-1;
\end{aligned}\tag{2}$$

$$p_0+\sum_{k=1}^{n}\sum_{i=0}^{k}p_{k(k-i,i)}+\sum_{k=n+1}^{m}\sum_{i=0}^{n}p_{k(n-i,i)}=1,\tag{3}$$

where $\lambda_k=(m-k)\lambda,\ 0\le k\le m-1.$

Introducing the notation

$$\alpha_k=\frac{\lambda_k}{\mu_1},\quad \alpha_{k2}=\frac{\lambda_k}{\mu_2},\quad 0\le k\le m-1;\qquad \eta=\frac{\mu_2}{\mu_1};$$

$$\tilde p_{k(ij)}=\frac{p_{k(ij)}}{p_0},\ 1\le k\le m;\qquad \tilde p_{k(k0)}=q_k,\ 1\le k\le n-1;\qquad \tilde p_{k(n0)}=q_k,\ n\le k\le m-1,$$

and using equations (1), we find:

$$\begin{aligned}
&\tilde p_{1(01)}=\alpha_{02};\quad \tilde p_{m(n0)}=\frac{\alpha_{m-1}}{n}q_{m-1};\quad \tilde p_{2(11)}=\frac{1}{\eta}\left((\alpha_1+1)q_1-\alpha_0\right);\\
&\tilde p_{k(k-1,1)}=\frac{1}{\eta}\left((\alpha_{k-1}+k-1)q_{k-1}-\alpha_{k-2}q_{k-2}\right),\quad 3\le k\le n;\\
&\tilde p_{k(k-j,j)}=\frac{1}{j\eta}\left(\left(\alpha_{k-1}+(j-1)\eta+k-j\right)\tilde p_{k-1(k-j,j-1)}-\alpha_{k-2}\tilde p_{k-2(k-j-1,j-1)}\right)-\\
&\qquad-\frac{k-j+1}{j\eta}\tilde p_{k-1(k-j+1,j-2)},\quad 2\le j\le n-1,\quad j+1\le k\le n;\\
&\tilde p_{k(0k)}=\frac{1}{k\eta}\left(\left(\alpha_{k-1}+(k-1)\eta\right)\tilde p_{k-1(0,k-1)}-\tilde p_{k-1(1,k-2)}\right),\quad 2\le k\le n;\\
&\tilde p_{n+1(n-1,1)}=\frac{1}{\eta}\left((\alpha_n+n)q_n-\alpha_{n-1}q_{n-1}\right);\\
&\tilde p_{n+1(n-j,j)}=\frac{1}{j\eta}\left(\left(\alpha_n+(j-1)\eta+n+1-j\right)\tilde p_{n(n-j+1,j-1)}-\alpha_{n-1}\tilde p_{n-1(n-j,j-1)}\right)-\\
&\qquad-\frac{n-j+2}{j\eta}\tilde p_{n(n-j+2,j-2)},\quad 2\le j\le n;\\
&\tilde p_{k(n-1,1)}=\frac{1}{\eta}\left((\alpha_{k-1}+n)q_{k-1}-\alpha_{k-2}q_{k-2}\right),\quad n+2\le k\le m;
\end{aligned}\tag{4}$$

$$\tilde{p}_{k(n-j,j)} = \frac{1}{j\eta}\Big(\big(\alpha_{k-1} + (j-1)\eta + n + 1 - j\big)\,\tilde{p}_{k-1(n-j+1,j-1)} - \alpha_{k-2}\tilde{p}_{k-2(n-j+1,j-1)}\Big) - \\ - \frac{n-j+2}{j\eta}\tilde{p}_{k-1(n-j+2,j-2)},\;\; 2 \le j \le n,\;\; n+2 \le k \le m.$$

Recurrence relations (4) allow us to calculate $\tilde{p}_{k(ij)}$ as linear functions of the unknown parameters q_k $(1 \le k \le m-1)$ in the following sequence:

$\tilde{p}_{m(n0)}$; $\tilde{p}_{k(k-1,1)}$ $(1 \le k \le n)$; $\tilde{p}_{k(k-2,2)}$ $(2 \le k \le n)$; $\tilde{p}_{k(k-3,3)}$ $(3 \le k \le n)$;
$\tilde{p}_{k(k-4,4)}$ $(4 \le k \le n)$;…; $\tilde{p}_{n-2(0,n-2)}$, $\tilde{p}_{n-1(1,n-2)}$, $\tilde{p}_{n(2,n-2)}$; $\tilde{p}_{n-1(0,n-1)}$, $\tilde{p}_{n(1,n-1)}$; $\tilde{p}_{n-1(1,n-2)}$, $\tilde{p}_{n(0n)}$;
$\tilde{p}_{n+1(n-j,j)}$, $(1 \le j \le n)$; $\tilde{p}_{k(n-1,1)}$ $(n+2 \le k \le m)$;
$\tilde{p}_{k(n-2,2)}$ $(n+2 \le k \le m)$; $\tilde{p}_{k(n-3,3)}$ $(n+2 \le k \le m)$; … ; $\tilde{p}_{k(0n)}$ $(n+2 \le k \le m)$.

To determine q_k $(1 \le k \le m-1)$, any $m-1$ equations from formulas (2) can be used. The equations (2) have not been involved in obtaining relations (4).

Using the normalization condition (3), we find steady-state probabilities by the formulas

$$p_0 = \left(1 + \sum_{k=1}^{n}\tilde{p}_{k(k0)} + \sum_{i=1}^{n}\sum_{k=i}^{n}\tilde{p}_{k(k-i,i)} + \sum_{k=n+1}^{m}\sum_{i=0}^{n}\tilde{p}_{k(n-i,i)}\right)^{-1},\quad p_k = p_0\tilde{p}_k,\quad 1 \le k \le m;$$

$$\tilde{p}_k = \sum_{i=0}^{k}\tilde{p}_{k(k-i,i)},\;\; 1 \le k \le n;\qquad \tilde{p}_k = \sum_{i=0}^{n}\tilde{p}_{k(n-i,i)},\;\; n+1 \le k \le m.$$

Here p_k is the steady-state probability that $n_c = k$. We calculate the steady-state characteristics - the average number of customers in the system $\mathbf{E}(n_c)$, the average queue length $\mathbf{E}(Q)$ and the average waiting time $\mathbf{E}(W)$ - by the formulas

$$\mathbf{E}(n_c) = \sum_{k=1}^{m} k p_k,\qquad \mathbf{E}(Q) = \sum_{k=n+1}^{m}(k-n)p_k,\qquad \mathbf{E}(W) = \frac{\mathbf{E}(Q)}{\lambda_{\text{av}}}.$$

Here λ_{av} is a steady-state value of the arrival rate of customers, defined by the equality

$$\lambda_{\text{av}} = \lambda\sum_{k=0}^{m-1}(m-k)p_k.$$

The parameter λ_{av} is a characteristic of the system capacity, because for the steady-state regime, we have the equality of the intensities of flows of customers arriving and served.

3. Numerical examples

Consider twentieth-channel closed queueing systems with an exponential distribution of the time generation of service requests and Erlangian service times of the second order for the following values of the parameters: $m = 30; 40; 50$; $\lambda = 1$; $\mu_1 = 1$, $\mu_2 = 2$.

The values of the steady-state probabilities characteristics of the system, found using the recurrence relations obtained in this paper, are presented in Table 1 and 2. In order to verify the obtained values, the tables contain the computing results evaluated by the GPSS World simulation system [6] for the time value $t = 10^6$.

Table 1

Stationary distribution of the number of customers in the system

k	Values of the steady-state probabilities p_k					
	$m = 30$		$m = 40$		$m = 50$	
	Recurrence method	GPSS World	Recurrence method	GPSS World	Recurrence method	GPSS World
0	$1.056\cdot10^{-12}$	0.000000	$1.496\cdot10^{-17}$	0.000000	$3.542\cdot10^{-24}$	0.000000
1	$4.760\cdot10^{-11}$	0.000000	$9.159\cdot10^{-16}$	0.000000	$2.811\cdot10^{-22}$	0.000000
2	$1.037\cdot10^{-9}$	0.000000	$2.736\cdot10^{-14}$	0.000000	$1.096\cdot10^{-20}$	0.000000
3	$1.454\cdot10^{-8}$	0.000000	$5.315\cdot10^{-13}$	0.000000	$2.799\cdot10^{-19}$	0.000000
4	$1.475\cdot10^{-7}$	0.000000	$7.548\cdot10^{-12}$	0.000000	$5.266\cdot10^{-18}$	0.000000
5	$1.153\cdot10^{-6}$	0.000001	$8.353\cdot10^{-11}$	0.000000	$7.784\cdot10^{-17}$	0.000000
10	0.001868	0.001883	$9.460\cdot10^{-7}$	0.000004	$4.320\cdot10^{-12}$	0.000000
15	0.074600	0.074590	0.000425	0.000406	$1.300\cdot10^{-8}$	0.000000
20	0.116516	0.117011	0.016895	0.016916	$5.467\cdot10^{-6}$	0.000002
21	0.086435	0.085993	0.027838	0.027924	0.000015	0.000015
22	0.056151	0.055804	0.042782	0.043157	0.0000411	0.000035
23	0.031662	0.031370	0.061267	0.060925	0.000106	0.000105
24	0.015288	0.015470	0.081565	0.081926	0.000258	0.000283
25	0.006201	0.006080	0.100634	0.100669	0.000596	0.000616
26	0.002057	0.002009	0.114632	0.114369	0.001308	0.001281
27	0.000536	0.000510	0.120022	0.119740	0.002715	0.002826
28	0.000103	0.000100	0.114914	0.115003	0.005327	0.005292
29	0.000013	0.000009	0.100011	0.099971	0.009856	0.009873
30	$7.985\cdot10^{-7}$	0.000000	0.078564	0.078377	0.017158	0.017189
31	-	-	0.055242	0.054938	0.028042	0.027858
32	-	-	0.034419	0.034723	0.042909	0.043099
33	-	-	0.018766	0.018799	0.061298	0.061235
34	-	-	0.008813	0.008854	0.081490	0.081228
35	-	-	0.003492	0.003495	0.100453	0.100511
36	-	-	0.001135	0.001104	0.114361	0.114362
37	-	-	0.000290	0.000314	0.119694	0.119866
38	-	-	0.000055	0.000046	0.114571	0.114588
39	-	-	$6.802\cdot10^{-6}$	0.000006	0.099694	0.099532

40	-	-	$4.149 \cdot 10^{-7}$	0.000000	0.078304	0.078887
42	-	-	-	-	0.034299	0.034318
44	-	-	-	-	0.008781	0.008891
46	-	-	-	-	0.001131	0.001142
48	-	-	-	-	0.000055	0.000062
50	-	-	-	-	$4.133 \cdot 10^{-7}$	0.000000

Table 2

Stationary characteristics of the system

Method	m	$\mathbf{E}(n_c)$	$\mathbf{E}(Q)$	$\mathbf{E}(W)$	λ_{av}
Recurrence	30	18.161167	0.402916	0.034033	11.838833
GPSS World	30	18.158	0.402	0.034	-
Recurrence	40	26.689706	6.724264	0.505193	13.310294
GPSS World	40	26.687	6.725	0.505	-
Recurrence	50	36.666670	16.666673	1.250001	13.333331
GPSS World	50	36.665	16.667	1.250	-

4. Conclusions

The numerical algorithm for solving a system of linear algebraic equations for the steady-state probabilities, proposed in this paper, is constructed taking into account the structural features of the system, in particular the presence of three or four unknown features in most of its equations. The obtained recurrence relations are used for the direct calculation of the solutions of the system, that allows us to reduce the amount of calculations in comparison with the case of application of one of the classical methods (direct or iterative). Using the obtained recurrence relations makes it possible to reduce the number of solved equations from $(n+1)(2m+2-n)/2$ to $m-1$.

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