JAM CM Journal of Applied Mathematics and Computational Mechanics 2017, 16(2), 159-171
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2017.2.13 e-ISSN 2353-0588

FIBONACCI-LIKE SEQUENCE ASSOCIATED WITH *K*-PELL, *K*-PELL-LUCAS AND MODIFIED *K*-PELL SEQUENCES

***Arfat Ahmad Wani* [1], *Sajad Ahmad Bhat* [2], *G.P.S. Rathore* [3]**

[1] *School of Studies in Mathematics, Vikram University Ujjain, Ujjain, India*
[2] *Department of Mathematics, Aligarh Muslim University, India*
[3] *Department of Statistics, Horticulture College Mandsaur, Mandsaur India*
arfatahmadwani@gmail.com, bhats6330@gmail.com, gps_rathore20@yahoo.co.in

Received: 24 February 2017; accepted: 19 June 2017

Abstract. Fibonacci sequence is a well known example of second order linear recurrence relatio. Besides Fibonacci numbers and their generalizations have many applications as well as interesting properties almost in every field of science such as in Physics, Biology, Mathematics (Algebra, Geometry and Number Theory itself). The main aim of the present article to introduce a generalization of Fibonacci sequence which is similar to *k*-Pell, *k*-Pell-Lucas, Modified k-Pell sequences and known as Fibonacci-Like sequence. After that we obtain some fundamental properties of Fibonacci-Like sequence such as Binet formulae of Fibonacci-Like sequence, binomial transform of the Fibonacci-Like sequence and sum of Fibonacci-Like numbers with indexes in an arithmetic sequence. In addition to this we obtain some new relations among *k*-Pell, *k*-Pell-Lucas, Modified *k*-Pell and Fibonacci-Like sequences.

MSC 2010: *11B37, 11B39, 11B83*
Keywords: *k-Pell sequence, k-Pell-Lucas sequence, Modified k-Pell sequence, generalized Fibonacci sequence*

1. Introduction

The classical Fibonacci numbers and their generalizations have very important properties and applications to almost every field of science and art. The Fibonacci sequence is defined by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ F_0 = 0,\ F_1 = 1 \tag{1}$$

Some sequences, such as Pell sequences, have a similar structure with the Fibonacci sequence, see for instance [1, 2]. In [3] Halici and Dasdemir presented some relations among Pell, Pell-Lucas and Modified Pell sequences. Halici in [4] introduced sums formulae for products of Pell $\{P_n\}$, Pell-Lucas $\{Q_n\}$ and Modified Pell

$\{q_n\}$ sequences. In [5-7] the authors delineated Binet's formula, generating and some other identities for Modified *k*-Pell numbers, *k*-Pell-Lucas numbers and *k*-Pell numbers respectively. In [8, 9] the authors acquainted identities of generalized Fibonacci sequences with Jacobsthal and Jacobsthal-Lucas sequences. Jhala et al. [10] obtained some properties for *k*-Jacobsthal numbers. In [11] Campos et al. presented the properties for *k*-Jacobsthal-Lucas sequence.

In [12, 13] the authors presented some properties for *k*-Jacobsthal and *k*-Lucas numbers with arithmetic indexes, say $an + r$, for fixed integers *a* and *r* respectively. In [14, 15] the authors employed several class of transforms like binomial, *k*-binomial, rising and falling transforms to the *k*-Lucas and *k*-Fibonacci sequences respectively and investigated the properties between the so obtained new sequences and *k*-Lucas or *k*-Fibonacci sequences.

2. *k*-Pell, *k*-Pell-Lucas and Modified *k*-Pell sequences

Definition 1 [7]
For any positive real number k, the k-Pell sequence say $\{P_{k,n}\}$ *is defined recurrently by*

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2},\ n \geq 2,\ P_{k,0} = 0,\ P_{k,1} = 1 \tag{2}$$

Definition 2 [6]
For any positive real number k, the k-Pell-Lucas sequence say $\{Q_{k,n}\}$ *is defined recurrently by*

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2},\ n \geq 2,\ Q_{k,0} = 2,\ Q_{k,1} = 2 \tag{3}$$

Definition 3 [5]
For any positive real number k, the k-Modified Pell sequence say $\{q_{k,n}\}$ *is defined recurrently by*

$$q_{k,n} = 2q_{k,n-1} + kq_{k,n-2},\ n \geq 2,\ q_{k,0} = 1,\ q_{k,1} = 1 \tag{4}$$

The same characteristic equation, associated to the recurrence relations (2), (3) and (4) is given by

$$x^2 - 2x - k = 0 \tag{5}$$

with two distinct roots a and b. Note that the roots of the equation (5) are

$$a = 1 + \sqrt{1+k} \ \text{ and } \ b = 1 - \sqrt{1+k}$$

The general forms or Binet formulae for the *k*-Pell sequence, *k*-Pell-Lucas sequence and Modified *k*-Pell sequence are given respectively:

$$P_{k,n} = \frac{a^n - b^n}{a - b} \tag{6}$$

$$Q_{k,n} = a^n + b^n \tag{7}$$

and

$$q_{k,n} = \frac{a^n + b^n}{2} \tag{8}$$

3. Fibonacci-Like sequence

Definition 4.
For any positive real number k, Fibonacci-Like say $\{R_{k,n}\}$ *is defined recurrently by*

$$R_{k,n} = 2R_{k,n-1} + kR_{k,n-2},\ n \geq 2,\ R_{k,0} = 2,\ R_{k,1} = 1 \tag{9}$$

Clearly $x^2 - 2x - k = 0$ is also the characteristic equation of the recurrence (9) and a and b are its two roots.
where $a = 1 + \sqrt{1+k}$ and $b = 1 - \sqrt{1+k}$
Here a and b have the following properties:
1. $a + b = 2$ and $a - 1 = 1 - b$
2. $ab = -k$
3. $(a-1)(b-1) = -(k+1)$
4. $a^2 = 2a + k$ and $b^2 = 2b + k$
5. $(a^2-1)(b^2-1) = k^2 - 2k - 3$
6. $a^2 + b^2 = 4 + 2k$

4. Binet formulae for Fibonacci-Like fequence

In this section we present explicit formulae for the n^{th} Fibonacci-Like number. In other words also we find the relations of Fibonacci-Like sequence with the *k*-Pell sequence and *k*-Pell-Lucas sequence and so, for this we have the following theorem:

Theorem 1

$$R_{k,n} = a^n + b^n - \frac{a^n - b^n}{a - b} = Q_{k,n} - P_{k,n},\ n \geq 0 \tag{10}$$

$$R_{k,n} = \frac{2\left(a^{n+1} - b^{n+1}\right) - 3\left(a^{n} - b^{n}\right)}{a-b} = 2P_{k,n+1} - 3P_{k,n},\ n \ge 0 \tag{11}$$

Proof. The general form of the Fibonacci-Like sequence may be expressed in the form:

$$R_{k,n} = Aa^{n} + Bb^{n} \tag{12}$$

where A and B are constants these can be determined by the initial conditions of the recurrence (9). So put the values $n = 0$ and $n = 1$ in the equation (12), we get

$$A + B = 2 \text{ and } Aa + Bb = 1$$

After solving the above system of equations for A and B, we get

$$A = \frac{1-2b}{a-b} \text{ and } B = \frac{2a-1}{a-b} \tag{13}$$

Therefore,

$$R_{k,n} = \frac{1}{a-b}\left[a^{n}(1-2b) + b^{n}(2a-1)\right] = \frac{1}{a-b}\left(a^{n} - 2ba^{n} - b^{n} + 2ab^{n}\right) \tag{14}$$

$$R_{k,n} = \frac{1}{a-b}\left(a^{n} - ba^{n} - ba^{n} - b^{n} + ab^{n} + ab^{n}\right)$$

$$R_{k,n} = \frac{1}{a-b}\left[a^{n}(1-b) - b^{n}(1-a) - ba^{n} + ab^{n}\right]$$

$$R_{k,n} = \frac{1}{a-b}\left[a^{n}(a-1) - b^{n}(b-1) - ba^{n} + ab^{n}\right]$$

$$R_{k,n} = \frac{1}{a-b}\left(a^{n}a - a^{n} - b^{n}b + b^{n} - ba^{n} + ab^{n}\right)$$

$$R_{k,n} = \frac{1}{a-b}\left[a^{n}(a-b) + b^{n}(a-b) - a^{n} + b^{n}\right]$$

$$R_{k,n} = \frac{1}{a-b}\left[\left(a^{n} + b^{n}\right)(a-b) - \left(a^{n} - b^{n}\right)\right] = a^{n} + b^{n} - \frac{a^{n} - b^{n}}{a-b}$$

This proves the first part of the theorem (10).
If we consult equation (14), we have

$$R_{k,n} = \frac{1}{a-b}\left(a^n - 2ba^n - b^n + 2ab^n\right)$$

$$R_{k,n} = \frac{1}{a-b}\left[a^n - 2\left(2-a\right)a^n - b^n + 2\left(2-b\right)b^n\right]$$

$$R_{k,n} = \frac{1}{a-b}\left(a^n - 4a^n + 2a^{n+1} - b^n + 4b^n - 2b^{n+1}\right)$$

$$R_{k,n} = \frac{1}{a-b}\left[-3a^n + 3b^n + 2\left(a^{n+1} - b^{n+1}\right)\right]$$

$$R_{k,n} = \frac{2\left(a^{n+1} - b^{n+1}\right) - 3\left(a^n - b^n\right)}{a-b}$$

This proves the second part of the theorem (10).
But for the sake of simplicity we can write the Binet formula of $\{R_{k,n}\}$ as

$$R_{k,n} = Aa^n + Bb^n \tag{15}$$

Here,

- $A = \dfrac{1-2b}{a-b}$ and $B = \dfrac{2a-1}{a-b}$
- $A + B = 2$
- $A - B = \dfrac{-2}{a-b}$
- $AB = \dfrac{3+4k}{\left(a-b\right)^2}$

Now if we interchange the positions A and B in the equation (12), we get another sequence say $\{R^*_{k,n}\}$ and is given by

$$R^*_{k,n} = Ba^n + Ab^n$$

$$R^*_{k,n} = Ab^n + Ba^n \tag{16}$$

But here our motive is to express $R^*_{k,n}$ in terms of *k*-Pell sequence, put the values of A and B from the equation (13) in the equation (16), we get

$$R^*_{k,n} = \frac{1}{a-b}\left[\left(1-2b\right)b^n + \left(2a-1\right)a^n\right]$$

$$R^*_{k,n} = \frac{1}{a-b}\left(b^n - 2b^{n+1} + 2a^{n+1} - a^n\right)$$

$$R^*_{k,n} = \frac{1}{a-b}\left[\, 2\left(a^{n+1} - b^{n+1}\right) - \left(a^n - b^n\right) \,\right]$$

$$R^*_{k,n} = 2P_{k,n+1} - P_{k,n} \tag{17}$$

Furthermore the relation between $\langle R_{k,n} \rangle$ and $\langle R^*_{k,n} \rangle$ is given by the following equation:

$$R_{k,n} = R^*_{k,n} - 2P_{k,n} \tag{18}$$

Lemma 1.

$$\frac{a^n - b^n}{a-b} = \frac{2R_{k,n+1} - R_{k,n}}{(3+4k)}, \; n \geq 0 \tag{19}$$

Proof. By the Binet formula of Fibonacci-Like sequence, we have

$$2R_{k,n+1} - R_{k,n} = 2\left(Aa^{n+1} + Bb^{n+1}\right) - \left(Aa^n + Bb^n\right)$$
$$= 2Aa^{n+1} - Aa^n + 2Bb^{n+1} - Bb^n = Aa^n(2a-1) + Bb^n(2b-1)$$

Since, $A(2a-1) = \dfrac{3+4k}{a-b}$ and $B(2b-1) = -\dfrac{3+4k}{a-b}$

Therefore,

$$2R_{k,n+1} - R_{k,n} = \frac{(3+4k)a^n - (3+4k)b^n}{a-b} = (3+4k)\frac{a^n - b^n}{a-b}$$
$$\frac{a^n - b^n}{a-b} = \frac{2R_{k,n+1} - R_{k,n}}{(3+4k)}$$

Theorem 2. (Catalan's Identity)

$$R_{k,n+r}\,R_{k,n-r} - R^2_{k,n} = \frac{(-k)^{n-r}}{3+4k}\left(2R_{k,r+1} - R_{k,r}\right)^2, \; 0 \leq r \leq n \tag{20}$$

Proof.

$$R_{k,n+r}\,R_{k,n-r} - R^2_{k,n} = \left(Aa^{n+r} + Bb^{n+r}\right)\left(Aa^{n-r} + Bb^{n-r}\right) - \left(Aa^n + Bb^n\right)^2$$
$$R_{k,n+r}\,R_{k,n-r} - R^2_{k,n} = A^2a^{2n} + ABa^{n+r}b^{n-r} + ABa^{n-r}b^{n+r} + B^2b^{2n} - A^2a^{2n}$$
$$- B^2b^{2n} - 2ABa^nb^n$$

$$R_{k,n+r}\,R_{k,n-r}-R_{k,n}^2=AB\left(a^{n+r}b^{n-r}+a^{n-r}b^{n+r}-2a^nb^n\right)=AB\left[(ab)^{n-r}\left(a^{2r}+b^{2r}\right)-2(ab)^n\right]$$

Since, $AB=\dfrac{3+4k}{(a-b)^2}$

$$R_{k,n+r}\,R_{k,n-r}-R_{k,n}^2=\frac{(3+4k)(-k)^{n-r}}{(a-b)^2}\left[a^{2r}+b^{2r}-2(ab)^r\right]$$

$$R_{k,n+r}\,R_{k,n-r}-R_{k,n}^2=(3+4k)\,(-k)^{n-r}\left(\frac{a^r-b^r}{a-b}\right)^2$$

Now by lemma (10), we have

$$R_{k,n+r}\,R_{k,n-r}-R_{k,n}^2=(3+4k)\,(-k)^{n-r}\left(\frac{R_{k,r+1}-R_{k,r}}{3+4k}\right)^2=\frac{(-k)^{n-r}}{3+4k}\left(2R_{k,r+1}-R_{k,r}\right)^2$$

Hence the result.

5. Relationships of Fibonacci-Like sequence with the other sequences

In this section we present some results on the relations of Fibonacci-Like sequence with the k-Pell sequence, k-Pell-Lucas sequence and k-Modified Pell sequence by simply using the Binet formulae of the said sequences.

Theorem 3.

$$R_{k,n+1}\,Q_{k,n}-Q_{k,n+1}\,R_{k,n}=-2(-k)^n\,,\;n\geq 0 \tag{21}$$

Proof.

$$R_{k,n+1}\,Q_{k,n}-Q_{k,n+1}\,R_{k,n}=\left(Aa^{n+1}+Bb^{n+1}\right)\left(a^n+b^n\right)-\left(a^{n+1}+b^{n+1}\right)\left(Aa^n+Bb^n\right)$$

$$R_{k,n+1}\,Q_{k,n}-Q_{k,n+1}\,R_{k,n}=Aa^{2n+1}+Aa^{n+1}b^n+Bb^{n+1}a^n+Bb^{2n+1}-Aa^{2n+1}$$
$$-Ba^{n+1}b^n-Ab^{n+1}a^n-Bb^{2n+1}$$

$$R_{k,n+1}\,Q_{k,n}-Q_{k,n+1}\,R_{k,n}=Aa^{n+1}b^n+Bb^{n+1}a^n-Ba^{n+1}b^n-Ab^{n+1}a^n$$

$$R_{k,n+1}\,Q_{k,n}-Q_{k,n+1}\,R_{k,n}=A\left(a^{n+1}b^n-b^{n+1}a^n\right)+B\left(b^{n+1}a^n-a^{n+1}b^n\right)$$

$$R_{k,n+1}\,Q_{k,n}-Q_{k,n+1}\,R_{k,n}=\left(a^nb^n\right)(a-b)(A-B)=(-k)^n\,(a-b)\,\frac{-2}{a-b}=-2(-k)^n$$

Theorem 4.

$$R_{k,n+1}\,P_{k,n}-P_{k,n+1}\,R_{k,n}=-2(-k)^n\,,\;n\geq 0 \tag{22}$$

Proof.

$$R_{k,n+1}\,P_{k,n}-P_{k,n+1}\,R_{k,n}=\left(Aa^{n+1}+Bb^{n+1}\right)\left(\frac{a^n-b^n}{a-b}\right)-\left(\frac{a^{n+1}-b^{n+1}}{a-b}\right)\left(Aa^n+Bb^n\right)$$

$$R_{k,n+1}\,P_{k,n}-P_{k,n+1}\,R_{k,n}=\frac{1}{a-b}\left[\begin{array}{l}\left(Aa^{n+1}+Bb^{n+1}\right)\left(a^n-b^n\right)-\left(a^{n+1}-b^{n+1}\right)\\ \left(Aa^n+Bb^n\right)\end{array}\right]$$

$$R_{k,n+1}\,P_{k,n}-P_{k,n+1}\,R_{k,n}=\frac{1}{a-b}\left(\begin{array}{l}Aa^{n+1}-Aa^{n+1}b^n+Bb^{n+1}a^n-Bb^{2n+1}-Aa^{n+1}\\ -Ba^{n+1}b^n+Aa^nb^{n+1}-Bb^{2n+1}\end{array}\right)$$

$$R_{k,n+1}\,P_{k,n}-P_{k,n+1}\,R_{k,n}=\frac{1}{a-b}\left(Aa^nb^{n+1}-Aa^{n+1}b^n+Bb^{n+1}a^n-Ba^{n+1}b^n\right)$$

$$R_{k,n+1}\,P_{k,n}-P_{k,n+1}\,R_{k,n}=-\frac{a-b}{a-b}(ab)^n\left(A+B\right)=-2(-k)^n$$

Theorem 5.

$$R_{k,n+1}\,q_{k,n}-q_{k,n+1}\,R_{k,n}=-\left(-k\right)^n,\ n\geq 0 \tag{23}$$

Proof.

$$R_{k,n+1}\,q_{k,n}-q_{k,n+1}\,R_{k,n}=\left(Aa^{n+1}+Bb^{n+1}\right)\left(\frac{a^n+b^n}{2}\right)-\left(\frac{a^{n+1}+b^{n+1}}{2}\right)\left(Aa^n+Bb^n\right)$$

$$R_{k,n+1}\,q_{k,n}-q_{k,n+1}\,R_{k,n}=\frac{1}{2}\left(\begin{array}{l}Aa^{2n+1}+Aa^{n+1}b^n+Bb^{n+1}a^n+Bb^{2n+1}-Aa^{n+1}-Ba^{n+1}b^n\\ -Aa^nb^{n+1}-Bb^{2n+1}\end{array}\right)$$

$$R_{k,n+1}\,q_{k,n}-q_{k,n+1}\,R_{k,n}=\frac{1}{2}\left(Aa^{n+1}b^n-Ab^{n+1}a^n+Bb^{n+1}a^n-Ba^{n+1}b^n\right)$$

$$R_{k,n+1}\,q_{k,n}-q_{k,n+1}\,R_{k,n}=\frac{a-b}{2}(ab)^n\left(A-B\right)=\frac{\left(a-b\right)}{2}(ab)^n\,\frac{-2}{a-b}$$

$$R_{k,n+1}\,q_{k,n}-q_{k,n+1}\,R_{k,n}=-\left(-k\right)^n$$

6. Binomial transform of the Fibonacci-Like sequence

In this section we introduce binomial transform of the Fibonacci-Like sequence $\{R_{k,n}\}$. Furthermore we present Binet formula and generating function of the binomial transform.

Given an integer sequence $A'=\{a_0,a_1,a_2,a_3,\cdots\}$, define the binomial transform B' of the sequence A' to be the sequence $B'\left(A'\right)=\left\{b_n\right\}$, where b_n is given by

$$b_n=\sum_{i=0}^{n}\binom{n}{i}a_i \tag{24}$$

Definition 5.
The binomial transform of the Fibonacci-Like sequence is denoted by $B'_k = \{b_{k,n}\}$ where:

$$b_{k,n} = \sum_{i=0}^{n} \binom{n}{i} R_{k,i} \tag{25}$$

Lemma 2. *The binomial transform of the Fibonacci-Like sequence verifies the recurrence*

$$b_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} \left(R_{k,i} + R_{k,i+1}\right) \tag{26}$$

Proof.

$$b_{k,n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} R_{k,i} = R_{k,0} + \sum_{i=1}^{n+1} \binom{n+1}{i} R_{k,i}$$

Since $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$, then

$$b_{k,n+1} = R_{k,0} + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1}\right] R_{k,i} = R_{k,0} + \sum_{i=1}^{n+1} \binom{n}{i} R_{k,i} + \sum_{i=1}^{n+1} \binom{n}{i-1} R_{k,i}$$

$$b_{k,n+1} = R_{k,0} + \sum_{i=1}^{n} \binom{n}{i} R_{k,i} + \sum_{i=0}^{n} \binom{n}{i} R_{k,i+1} = \sum_{i=0}^{n} \binom{n}{i} R_{k,i} + \sum_{i=0}^{n} \binom{n}{i} R_{k,i+1}$$

$$b_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} \left(R_{k,i} + R_{k,i+1}\right)$$

Theorem 6. *Binomial transform of the Fibonacci-Like sequence,* $B'_k = \{b_{k,n}\}$ *verifies the recurrence relation*

$$b_{k,n+1} = 4b_{k,n} - (3-k)\, b_{k,n-1},\ b_{k,0} = 2,\ b_{k,1} = 3 \tag{27}$$

Proof. Since

$$b_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} \left(R_{k,i} + R_{k,i+1}\right)$$

$$b_{k,n+1} = R_{k,0} + R_{k,1} + \sum_{i=1}^{n} \binom{n}{i}\left(R_{k,i} + R_{k,i+1}\right) = R_{k,0} + R_{k,1} + \sum_{i=1}^{n} \binom{n}{i}\left(R_{k,i} + 2R_{k,i} + kR_{k,i-1}\right)$$

$$b_{k,n+1} = R_{k,0} + R_{k,1} + \sum_{i=1}^{n} \binom{n}{i}\left(3R_{k,i} + kR_{k,i-1}\right)$$

$$b_{k,n+1} = 3\sum_{i=1}^{n} \binom{n}{i} R_{k,i} + k\sum_{i=1}^{n} \binom{n}{i} R_{k,i-1} + R_{k,0} + R_{k,1}$$

$$b_{k,n+1} = 3\sum_{i=1}^{n} \binom{n}{i} R_{k,i} + 3R_{k,0} + k\sum_{i=1}^{n} \binom{n}{i} R_{k,i-1} - 3R_{k,0} + R_{k,0} + R_{k,1}$$

$$b_{k,n+1} = 3\sum_{i=0}^{n} \binom{n}{i} R_{k,i} + k\sum_{i=1}^{n} \binom{n}{i} R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n+1} = 3b_{k,n} + k\sum_{i=1}^{n} \binom{n}{i} R_{k,i-1} - 2R_{k,0} + R_{k,1} \tag{28}$$

On the other hand using the fact of $\binom{n-1}{n} = 0$ and putting $n-1$ instead of n in (28), we get

$$b_{k,n} = 3b_{k,n-1} + k\sum_{i=1}^{n-1} \binom{n-1}{i} R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n} = \left(2+1\right) b_{k,n-1} + k\sum_{i=1}^{n-1} \binom{n-1}{i} R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n} = 2b_{k,n-1} + \sum_{i=1}^{n-1} \binom{n-1}{i} R_{k,i} + k\sum_{i=1}^{n-1} \binom{n-1}{i} R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n} = 2b_{k,n-1} + \sum_{i=1}^{n} \binom{n-1}{i-1} R_{k,i-1} + k\sum_{i=1}^{n-1} \binom{n-1}{i} R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n} = 2b_{k,n-1} + \sum_{i=1}^{n} \binom{n-1}{i-1} R_{k,i-1} + k\left[\binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-1}{n-1} + \binom{n-1}{n}\right] R_{k,i-1}$$
$$- 2R_{k,0} + R_{k,1}$$

$$b_{k,n} = 2b_{k,n-1} + \sum_{i=1}^{n} \binom{n-1}{i-1} R_{k,i-1} + k\sum_{i=1}^{n} \binom{n-1}{i} R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n} = 2b_{k,n-1} + \sum_{i=1}^{n} \left[\binom{n-1}{i-1} + k\binom{n-1}{i}\right] R_{k,i-1} + -2R_{k,0} + R_{k,1}$$

$$b_{k,n} = 2b_{k,n-1} + \sum_{i=1}^{n} \left[\binom{n-1}{i-1} + k\binom{n-1}{i} + k\binom{n-1}{i-1} - k\binom{n-1}{i-1}\right] R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n} = 2b_{k,n-1} + \sum_{i=1}^{n} \left[\left(1-k\right)\binom{n-1}{i-1} + k\binom{n}{i}\right] R_{k,i-1} - 2R_{k,0} + R_{k,1}$$

$$b_{k,n}=2b_{k,n-1}+k\sum_{i=1}^{n}\binom{n}{i}R_{k,i-1}+(1-k)\sum_{i=1}^{n}\binom{n-1}{i-1}R_{k,i-1}-2R_{k,0}+R_{k,1}$$

$$b_{k,n}=2b_{k,n-1}+k\sum_{i=1}^{n}\binom{n}{i}R_{k,i-1}+(1-k)\sum_{i=1}^{n-1}\binom{n-1}{i-1}R_{k,i}-2R_{k,0}+R_{k,1}$$

$$b_{k,n}=2b_{k,n-1}+(1-k)b_{k,n-1}+k\sum_{i=1}^{n}\binom{n}{i}R_{k,i-1}-2R_{k,0}+R_{k,1}$$

$$b_{k,n}=(3-k)b_{k,n-1}+k\sum_{i=1}^{n}\binom{n}{i}R_{k,i-1}-2R_{k,0}+R_{k,1}$$

$$k\sum_{i=1}^{n}\binom{n}{i}R_{k,i-1}-2R_{k,0}+R_{k,1}=b_{k,n}-(3-k)b_{k,n-1}$$

Therefore from equation (28), we have

$$b_{k,n+1}=3b_{k,n}+b_{k,n}-(3-k)b_{k,n-1}$$
$$b_{k,n+1}=4b_{k,n}-(3-k)b_{k,n-1}$$

Hence the result.

7. Fibonacci-Like sequence with arithmetic indexes

In this section we present property of Fibonacci-Like sequence with indexes in an arithmetic sequence, say $mn+r$, for fixed integers *m* and $r, 0 \le r \le m-1$.

Theorem 7. The sum of Fibonacci-Like numbers of kind $mn+r$ is

$$\sum_{i=0}^{n}R_{k,mi+r}\,x^{n}=\frac{R_{k,m(n+1)+r}-(-k)^{m}R_{k,mn+r}-R_{k,r}+(-k)^{r}R^{*}_{k,m-r}}{Q_{k,m}-(-k)^{m}-1} \tag{29}$$

Proof.

$$\sum_{i=0}^{n}R_{k,mi+r}=\sum_{i=0}^{n}\left(Aa^{mi+r}+Bb^{mi+r}\right)=A\sum_{i=0}^{n}a^{mi+r}+B\sum_{i=0}^{n}b^{mi+r}$$

$$\sum_{i=0}^{n}R_{k,mi+r}=A\left(a^{r}+a^{m+r}+a^{2m+r}+\cdots+a^{mn+r}\right)+B\left(b^{r}+b^{m+r}+b^{2m+r}+\cdots+b^{mn+r}\right)$$

$$\sum_{i=0}^{n}R_{k,mi+r}=A\frac{a^{r}\left[a^{m(n+1)}-1\right]}{a^{m}-1}+B\frac{b^{r}\left[b^{m(n+1)}-1\right]}{b^{m}-1}=A\frac{a^{mn+m+r}-a^{r}}{a^{m}-1}+B\frac{b^{mn+m+r}-b^{r}}{b^{m}-1}$$

$$\sum_{i=0}^{n}R_{k,mi+r}=\frac{A\left(b^{m}-1\right)\left(a^{mn+m+r}-a^{r}\right)+B\left(a^{m}-1\right)\left(b^{mn+m+r}-b^{r}\right)}{\left(a^{m}-1\right)\left(b^{m}-1\right)}$$

$$\sum_{i=0}^{n} R_{k,mi+r} = \frac{A\left(b^m a^{mn+m+r} - b^m a^r - a^{mn+m+r} + a^r\right) + B\left(a^m b^{mn+m+r} - a^m b^r - b^{mn+m+r} + b^r\right)}{\left(a^m - 1\right)\left(b^m - 1\right)}$$

$$\sum_{i=0}^{n} R_{k,mi+r} = \frac{A\left(b^m a^m a^{mn+r} - b^m a^r - a^{mn+m+r} + a^r\right) + B\left(a^m b^m b^{mn+r} - a^m b^r - b^{mn+m+r} + b^r\right)}{(ab)^m - a^m - b^m + 1}$$

$$\sum_{i=0}^{n} R_{k,mi+r} = \frac{b^m a^m\left(Aa^{mn+r} + Bb^{mn+r}\right) - \left(Aa^{mn+m+r} + Bb^{mn+m+r}\right) + \left(Aa^r + Bb^r\right) - \left(Ab^m a^r + Ba^m b^r\right)}{(ab)^m - \left(a^m + b^m\right) + 1}$$

$$\sum_{i=0}^{n} R_{k,mi+r} = \frac{b^m a^m\left(Aa^{mn+r} + Bb^{mn+r}\right) - \left[Aa^{m(n+1)+r} + Bb^{m(n+1)+r}\right] + \left(Aa^r + Bb^r\right) - a^r b^r \left(Ab^{m-r} + Ba^{m-r}\right)}{(-k)^m - Q_{k,m} + 1}$$

$$\sum_{i=0}^{n} R_{k,mi+r} = \frac{(-k)^m R_{k,mn+r} - R_{k,m(n+1)+r} + R_{k,r} - (-k)^r R^*_{k,m-r}}{-Q_{k,m} + (-k)^m + 1}$$

$$\sum_{i=0}^{n} R_{k,mi+r} = \frac{R_{k,m(n+1)+r} - (-k)^m R_{k,mn+r} - R_{k,r} + (-k)^r R^*_{k,m-r}}{Q_{k,m} - (-k)^m - 1}$$

Hence the result.

8. Conclusion

As we know that Fibonacci numbers and their generalizations have practical applications in every field of science. In the present article first of all we introduced a Fibonacci-Like sequence and after that we obtained some properties about the Fibonacci-Like sequence and also obtained some relations among *k*-Pell, *k*-Pell-Lucas, Modified *k*-Pell and Fibonacci-Like sequences.

References

[1] Horadam A.F., Applications of modified Pell numbers to representations, Ulam Quarterly 1994, 3(1), 34-53.

[2] Horadam A.F., Pell identities, Fibonacci Quarterly 1971, 9(3), 245-252.

[3] Halici S., Daşdemir A., On some relationships among Pell, Pell-Lucas and modified Pell sequences, Fen Bilimleri Dergisi 2010, 1(1), 141-145.

[4] Halici S., Some sums formulae for products of terms of Pell, Pell-Lucas and modified Pell sequences, Fen Bilimleri Dergisi 2011, 151-155.

[5] Catarino P., Vasco P., Modified k-Pell sequence: Some identities and ordinary generating function, Applied Mathematica Sciences 2013, 7(121), 6031-6037.

[6] Catarino P., Vasco P., On some identities and generating functions for k-Pell-Lucas sequence, Applied Mathematical Sciences 2013, 7(98), 4867-4873.

[7] Catarino P., On some identities and generating functions for k-Pell numbers, International Journal of Mathematical Analysis 2013, 7(38), 1877-1884.

[8] Singh B., Bhatnagar S., Sikhwal O., Generalized identities of companion Fibonacci-Like sequences, Global Journal of Mathematical Analysis 2013, 1(3), 104-109.

[9] Gupta V.K., Panwar Y.K., Sikhwal O., Generalized Fibonacci sequences, Theoretical Mathematics and Applications 2012, 2(2), 115-124.

[10] Jhala D., Sisodiya K., Rathore G.P.S., On some identities for k-Jacobsthal numbers, Int. Journal of Math. Analysis 2013, 7(12), 551-556.

[11] Campos H., Catarino P., Aires A.P., Vasco P., Borges A., On some identities of k-Jacobsthal--Lucas numbers, Int. Journal of Math. Analysis 2014, 8(10), 489-494.

[12] Jhala D., Rathore G.P.S., Sisodiya K., Some properties of k-Jacobsthal numbers with arithmetic indexes, Turkish Journal of Analysis and Number Theory 2014, 2(4), 119-124.

[13] Singh B., Bhadouria P., Sikhwal O., Sum properties for the k-Lucas numbers with arithmetic indexes, Journal of Mathematical and Computational Science 2014, 4(1), 105-117.

[14] Bhadouria P., Jhala D., Singh B., Binomial transforms of the k-Lucas sequences and its properties, Journal of Mathematics and Computer Science 2014, 8, 81-92.

[15] Falcon S., Plaza A., Binomial transforms of the k-Fibonacci sequence, International Journal of Nonlinear Sciences and Numerical Simulation 2009, 10, 1527-1538.