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PROPERTIES OF ENTIRE SOLUTIONS OF SOME LINEAR PDE'S

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Abstract. In this paper, there are improved sufficient conditions of boundedness of the L-index in a direction for entire solutions of some linear partial differential equations. They are new even for the one-dimensional case and $L \equiv 1$. Also, we found a positive continuous function l such that entire solutions of the homogeneous linear differential equation with arbitrary fast growth have a bounded l-index and estimated its growth.

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1. Introduction

Let $L:\mathbb{C}^n \to \mathbb{R}_+ := (0,+\infty)$ be a continuous function. An entire function $F(z)$, $z\in\mathbb{C}^n$, is called [1-4] *a function of bounded L-index in a direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if there exists $m_0\in\mathbb{Z}_+$ such that

$$L^{-m}(z)\,|\,\partial_{\mathbf{b}}^m F(z)\,|/m! \le \max\left\{ L^{-k}(z)\,|\,\partial_{\mathbf{b}}^k F(z)\,|/k! \;:\; 0\le k\le m_0\right\} \qquad (1)$$

for every $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$, where $\partial_{\mathbf{b}}^0 F(z)=F(z)$, $\partial_{\mathbf{b}}^k F(z):=\partial_{\mathbf{b}}^1(\partial_{\mathbf{b}}^{k-1}F(z))$, $k\ge 2$, $\partial_{\mathbf{b}}^1 F(z):=\sum_{j=1}^n \frac{\partial F(z)}{\partial z_j}\overline{b_j} = \langle \mathbf{grad}\ F, \overline{\mathbf{b}}\rangle$. The least such integer $m_0=m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *of the entire function* $F(z)$ *and is denoted by* $N_{\mathbf{b}}(F,L)=m_0$. In the case $n=1$, $\mathbf{b}=1$, $L\equiv l$, $F\equiv f$ we obtain the definition of an entire function of one variable of bounded l-index (see [5, 6]). And the value of the l-index is denoted by $N(f,l)$.

This paper is devoted to three problems in theory of partial differential equations in $\mathbb{C}^n$ and differential equations in a complex plane.

At first, we consider the partial differential equation

$$g_0(z)\partial_{\mathbf{b}}^p w + g_1(z)\partial_{\mathbf{b}}^{p-1} w + \ldots + g_p(z)w = h(z), \tag{2}$$

where g_j, h are entire functions in $\mathbb{C}^n$, $j \in \{0,1,\ldots,p\}$. There are known sufficient conditions [1, 2, 4] of boundedness of the L-index in the direction for entire solutions of (2). In particular, some inequalities must be satisfied outside discs of any radius. Replacing the universal quantifier by the existential, we relax the conditions.

Also the ordinary differential equation

$$w^{(p)} = f(z,w). \tag{3}$$

is considered. Shah, Fricke, Sheremeta, Kuzyk [6-8] did not investigate an index boundedness of the entire solution of (3) because the right hand side of (3) is a function of two variables. But now in view of entire function theory of bounded L-index in direction, it is natural to pose and to consider the following question.

Problem 1 [3, Problem 4]. *Let* $p=1$, $f(z,w)$ *be a function of bounded* L*-index in directions* $(1,0)$ *and* $(0,1)$. *What is a function* l *such that an entire solution* $w=w(z)$ *of equation (3) has a bounded* l*-index?*

Finally, we consider the linear homogeneous differential equation of the form

$$f^{(p)} + g_1(z)f^{(p-1)} + \cdots + g_p(z)f = 0, \tag{4}$$

which is obtained from (2), if $n=1$, $\mathbf{b}=1$, $h(z)\equiv 0$, $g_0(z)\equiv 1$. There is a known result of Kuzyk and Sheremeta [5] about the growth of the entire function of the bounded l-index. Later Kuzyk, Sheremeta [6] and Bordulyak [9] investigated the boundedness of the l-index of entire solutions of equation (4) and its growth.

Meanwhile, many mathematicians such as Kinnunen, Heittokangas, Korhonen, Rättya, Cao, Chen, Yang, Hamani, Belaïdi [10-14] used the iterated orders to study the growth of solutions (4). Lin, Tu and Shi [15] proposed a more flexible scale to study the growth of solutions. They used $[p,q]$-order. But, the iterated orders and $[p,q]$-orders do not cover arbitrary growth (see example in [16]). There is considered a more general approach to describe the relations between the growth of entire coefficients and entire solutions of (4). In view of results from [16], the authors raise the question: *what is a positive continuous function* l *such that entire* *solutions of (4) with arbitrary fast growth have bounded l-index?* We provide an answer to the question.

2. Auxiliary propositions and notations

For $\eta>0$, $z\in\mathbb{C}^n$, $\mathbf{b}=\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and positive continuous function $L:\mathbb{C}^n\to\mathbb{R}_+$ we define $\lambda_1(\eta)=\lambda_1^{\mathbf{b}}(\eta):=\inf\limits_{z\in\mathbb{C}^n}\inf\{L(z+t\mathbf{b})/L(z):|t|\le\eta/L(z)\}$, $\lambda_2(\eta)=\lambda_2^{\mathbf{b}}(\eta):=\sup\limits_{z\in\mathbb{C}^n}\sup\{L(z+t\mathbf{b})/L(z):|t|\le\eta/L(z)\}$. By $Q_{\mathbf{b}}^n$ we denote a class of functions L, which satisfiy the condtion $(\forall\,\eta\ge 0):0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty$.

For simplicity, we also use a notation $Q=Q_1^1$.

Theorem A [1, 4]. *Let $L\in Q_{\mathbf{b}}^n$. An entire in $\mathbb{C}^n$ function F is of bounded L-index in direction $\mathbf{b}$ if and only if there exist numbers r_1 and r_2, $0<r_1<1<r_2<+\infty$, and $P_1\ge 1$ such that for all $z^0\in\mathbb{C}^n$*

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=r_2/L(z^0)\right\}\le P_1\max\left\{|F(z^0+t\mathbf{b})|:|t|=r_1/L(z_0)\right\}.$$

Let us to write $G_r(F):=G_r^{\mathbf{b}}(F)=\bigcup\limits_{z:F(z)=0}\{z+t\mathbf{b}:|t|<r/L(z)\}$, $n_{z^0}(r,F):=\sum_{|a_k^0|\le r}1$, a_k^0 - zeros of the function $F(z^0+t\mathbf{b})$ for a given $z^0\in\mathbb{C}^n$. If for all $t\in\mathbb{C}$ $F(z^0+t\mathbf{b})\equiv 0$, then we put $n_{z^0}(r)=-1$.

Theorem B [1, 4]. *Let F be an entire function of the bounded L-index in the direction $\mathbf{b}$, $L\in Q_{\mathbf{b}}^n$. Then for every $r>0$ and for every $m\in\mathbb{N}$ there exists $P=P(r,m)>1$ such that for all $z\in\mathbb{C}^n\setminus G_r(F)$ $\left|\partial_{\mathbf{b}}^m F(z)\right|\le PL^m(z)|F(z)|$.*

Theorem C [1, 4]. *Let F be an entire function in $\mathbb{C}^n$, $L\in Q_{\mathbf{b}}^n$. Then the function F is of bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$ if and only if the following conditions hold: 1) for every $R>0$ there exists $P=P(R)>0$ such that for each $z\in\mathbb{C}^n\setminus G_R(F)$ $\left|\partial_{\mathbf{b}}^1 F(z)\right|\le PL(z)F(z)$; 2) for every $r>0$ there exists $\tilde{n}(r)\in Z_+$ such that for every $z\in\mathbb{C}^n$ $n_z\left(r/L(z),F\right)\le\tilde{n}(r)$.*

Theorem D [1, 4]. *Let $L\in Q_{\mathbf{b}}^n$. An entire function $F(z)$ has a bounded L-index in direction $\mathbf{b}$ if and only if there exist $p\in\mathbb{Z}_+$ and $C>0$ such that $\left|L^{-p-1}(z)\partial_{\mathbf{b}}^{p+1}F(z)\right|\le C\max\left\{|L^{-k}(z)\partial_{\mathbf{b}}^k F(z)|:0\le k\le p\right\}$ for each $z\in\mathbb{C}^n$.*

Theorem E [17]. *Let G be a bounded closed domain in $\mathbb{C}$, $l:\mathbb{C}\to\mathbb{R}_+$ be a continuous function, f be an entire function. Then there exists $m_0\in\mathbb{N}$ such that for all $t\in G$ and for all $p\in\mathbb{N}$*

$$l^{-p}(t)\,|f^{(p)}(t)|\big/p!\le\max\left\{\, l^{-k}(t)\,|f^{(k)}(t)|\big/k!:\ 0\le k\le m_0\right\}.$$

Theorem F [5]. *Let l be a positive continuously differentiable function of real $t\in[0,+\infty)$. Suppose that $(-l'(t))^+=o(l^2(t))$ as $t\to+\infty$, where $a^+=\max\{a,0\}$. If an entire function f has a bounded l-index then* $\limsup\limits_{r\to+\infty}\ln M(r,f)\Big/\int_0^r l(t)dt\le N(f,l)+1.$

3. Boundedness of L-index in direction of entire solutions of some linear partial differential equations

Denote $g^*(z)=h(z)\cdot\prod_{j=0}^{p}g_j(z)$, $n(r,g^*)=\sup\limits_{z\in\mathbb{C}^n}n_{\mathbf{b}}(r/L(z),z,1/g^*)$, $n_{z^0}(r,F):=\sum_{|a_k^0|\le r}1$ $H(F):=\bigcup\limits_{\substack{z\in Z_F\\ \forall t\in\mathbb{C}\,F(z+t\mathbf{b})\equiv0}}\{z+t\mathbf{b}:t\in\mathbb{C}\}$, where Z_F is a zero set of the function F. The following theorem is valid.

Theorem 1. *Let $L\in Q_{\mathbf{b}}^n$, and $g_0(z),\dots,g_p(z),h(z)$ be entire functions of the bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$. Suppose that there exist $r\in(0;r^*)$ and $T>0$ such that for each $z\in\mathbb{C}^n\setminus G_r(g_0)$ and $j=1,\dots,p$*

$$|g_j(z)|\le TL^j(z)\,|g_0(z)|. \tag{5}$$

Then an entire function F satisfying (2) *has bounded L-index in the direction* $\mathbf{b}$.

Proof: Theorem C provides that $n(r,g^*)<+\infty$ and $r^*>0$. Denote $G_r=(G_r(h)\setminus H(h))\cup G_r(g_0)\cup\bigcup_{j=1}^{p}(G_r(g_j)\setminus H(g_j))$. Suppose that $\mathbb{C}^n\setminus G_r\ne\varnothing$. Theorem B and inequality (5) imply that there exist $r\in(0;r^*]$ and $T^*>0$ such that for all $z\in\mathbb{C}^n\setminus G_r$

$$\left|\partial_{\mathbf{b}}^1 h(z)\right|\le T^*\,|h(z)|\,L(z),\ \ |g_j(z)|\le T^*\,|g_0(z)|\,L^j(z),\ \ j\in\{1,2,\dots,p,\},$$

$$\left|\partial_{\mathbf{b}}^1 g_j(z)\right|\le P(r)L(z)\,|g_j(z)|\le T^*(r)\,|g_0(z)|\,L^{j+1}(z),\ j\in\{0,1,2,\dots,p,\}.$$

By equation (2), we evaluate the derivative in the direction $\mathbf{b}$:

$$g_0(z)\partial_{\mathbf{b}}^{p+1}F(z)+\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p+1-j}F(z)+\sum_{j=0}^{p}\partial_{\mathbf{b}}^{1}g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)=\partial_{\mathbf{b}}^{1}h(z).$$

The obtained equality implies that for all $z\in\mathbb{C}^n\setminus G_r$:

$$|g_0(z)\partial_{\mathbf{b}}^{p+1}F(z)|\le|\partial_{\mathbf{b}}^{1}h(z)|+\sum_{j=1}^{p}|g_j(z)\partial_{\mathbf{b}}^{p+1-j}F(z)|+\sum_{j=0}^{p}|\partial_{\mathbf{b}}^{1}g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)|\le$$

$\le T^*((T^*+1)(p+1)+p)|g_0(z)|L^{p+1}(z)\max_{0\le j\le p}\left\{L^{-j}(z)\left|\partial_{\mathbf{b}}^{j}F(z)\right|\right\}$. Thus, there exists $P_3>0$ such that for all $z\in\mathbb{C}^n\setminus G_r$

$$L^{-p-1}(z)\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\le P_3\max\left\{L^{-j}(z)\left|\partial_{\mathbf{b}}^{j}F(z)\right|:0\le j\le p\right\}. \tag{6}$$

If $z'\in A:=H(g_0)\setminus\bigcup_{j=1}^{p}(G_r(g_j)\setminus H(g_j))$ then there exists a sequence of points $z^m\in\mathbb{C}^n\setminus G_r$ satisfying (6) and such that $z^m\to z'$ with as $m\to\infty$. Substituting $z=z^m$ in (6) and taking the limit as $m\to\infty$ we obtain that this inequality is valid for all $z\in A\cup(\mathbb{C}^n\setminus G_r)$ If $\mathbb{C}^n=A\cup(\mathbb{C}^n\setminus G_r)$ (i.e. all zeros of g^* belong to $H(g^*)$) then by Theorem D the entire function satisfying (2) has a bounded $\mathbf{L}$-index in the direction $\mathbf{b}$. Otherwise, $n(s,g^*)\ge 1$. Since $r\in(0,r^*)$ and $r^*=\sup_{s\ge1}\frac{(s-1)\lambda_1(s)}{8(n(s,g^*)+1)}$ then there exists $r'\ge1$ such that $r\le\frac{(r'-1)\lambda_1(r')}{8(n(r',g^*)+1)}$. Let z^0 be an arbitrary point from $\mathbb{C}^n$ and $K^0=\{z^0+t\mathbf{b}:|t|\le r'/L(z^0)\}$. Since the entire functions $g_0,g_1,\dots,g_p,h$ have a bounded L-index in the direction $\mathbf{b}$, by Theorem C the set K^0 contains at most $n(r',g^*)$ zeros of the functions or $K^0\subset Z_{g^*}$. Let c_m^0 be zeros of the slice function g^* (i.e. $g^*(z^0+c_m^0\mathbf{b})=0$) such that $z^0+c_m^0\mathbf{b}\in K^0\cap((Z_h\setminus H(h))\cup\bigcup_{j=0}^{p}(Z_{g_j}\setminus H(g_j))$ where $m\in\mathbb{N}$, $m\le n(r',g^*)$. Since $L\in Q_{\mathbf{b}}^n$ we have $L(z^0+c_m^0\mathbf{b})\ge\lambda_1(r')L(z^0)$. Obviously,

$$\tilde{K}_m^0:=\left\{z^0+t\mathbf{b}:|t-c_m^0|\le\frac{r}{L(z^0+c_m^0\mathbf{b})}\right\}\subset K_m^0:=\left\{z^0+t\mathbf{b}:|t-c_m^0|\le\frac{r'-1}{8(n(r',g^*)+1)L(z^0)}\right\}.$$

Thus, if $z^0+t\mathbf{b}\in K^0\setminus\bigcup_{z^0+c_m^0\mathbf{b}\in K^0}K_m^0$, then (6) holds. Hence, for these points $z^0+t\mathbf{b}$ the inequality $L(z^0)\ge L(z^0+t\mathbf{b})/\lambda_2(r')$ and (6) imply

$$L^{-p-1}(z^0)\left|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})\right|\le(\lambda_2(r'))^{p+1}L^{-p-1}(z^0+t\mathbf{b})\,|\,\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|\le$$
$$\le P_3\left(\frac{\lambda_2(r')}{\lambda_1(r')}\right)^p\lambda_2(r')\max_{0\le j\le p}\left\{L^{-j}(z^0)\left|\partial_{\mathbf{b}}^jF(z^0+t\mathbf{b})\right|\right\}=P_4g_{z^0}(t), \tag{8}$$

where $P_4=P_3\lambda_2(r')\left(\frac{\lambda_2(r')}{\lambda_1(r')}\right)^p$ and $g_{z^0}(t)=\max\left\{L^{-j}(z^0)\left|\partial_{\mathbf{b}}^jF(z^0+t\mathbf{b})\right|:0\le j\le p\right\}$.

Let D be the sum of the diameters of K_m^0. Then $D<\frac{r'-1}{4L(z^0)}$. Therefore, there exist numbers $r_1\in[r'/4,r'/2]$ and $r_2\in[(3r'+1)/4,r']$ such that if $z^0+t\mathbf{b}\in C_j=\left\{z^0+t\mathbf{b}:|t|=r_j/L(z^0)\right\}$, $j\in\{1,2\}$, then $z^0+t\mathbf{b}\in K^0\setminus\bigcup_{c_m^0\in K^0}K_m^0$. We choose arbitrary points $z^0+t_1\mathbf{b}\in C_1$ and $z^0+t_2\mathbf{b}\in C_2$ and connect them by a smooth curve $\gamma=\{z^0+t\mathbf{b}:t=t(s),0\le s\le 1\}$ such that $F(z^0+t(s)\mathbf{b})\ne 0$ and $\gamma\subset K^0\setminus\bigcup_{c_m^0\in K^0}K_m^0$. This curve can be selected such that $|\gamma|<\frac{3|\mathbf{b}|r'}{L(z^0)}$. Then on γ inequality (7) holds. It is easy to prove that the function $g_{z^0}(t(s))$ is continuous on $[0,1]$ and continuously differentiable except a finite number of points. Moreover, for a complex-valued function of real variable the inequality $\frac{d}{ds}|\phi(s)|\le|\phi'(s)|$ holds except points, where $\varphi(s)=0$. Then, in view of (7), we have

$$\frac{d}{ds}g_{z^0}(t(s))\le\max_{0\le j\le p}\left\{\frac{d}{ds}L^{-j}(z^0)\left|\partial_{\mathbf{b}}^jF(z^0+t(s)\mathbf{b})\right|\right\}\le$$
$$\le\max_{0\le j\le p+1}\left\{L^{-j}(z^0)\left|\partial_{\mathbf{b}}^jF(z^0+t(s)\mathbf{b})\right|\right\}|t'(s)|\,L(z^0)\le P_5g_{z^0}(t(s))\,|t'(s)|\,L(z^0),$$

where $P_5=\max\{1,P_4\}$. Integrating over the variable s we deduce $\left|\ln\frac{g_{z^0}(t_2)}{g_{z^0}(t_1)}\right|=\left|\int_0^1\frac{1}{g_{z^0}(t(s))}\frac{d}{ds}g_{z^0}(t(s))ds\right|\le P_5L(z^0)\int_0^1|t'(s)|\,ds\le P_5L(z^0)|\gamma|\le 3|\mathbf{b}|r'P_5$, i.e. $g_{z^0}(t_2)\le g_{z^0}(t_1)\exp\{3|\mathbf{b}|r'P_5\}$. We can choose t_2 such that $|F(z^0+t_2\mathbf{b})|=\max\{|F(z^0+t\mathbf{b})|:z^0+t\mathbf{b}\in C_2\}$. Hence,

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{3r'+1}{4L(z^0)}\right\}\leq|F(z^0+t_2\mathbf{b})|\leq g_{z^0}(t_2)\leq g_{z^0}(t_1)\exp\{3|\mathbf{b}|r'P_5\}. \quad (9)$$

Since $z^0+t_1\mathbf{b}\in C_1=\left\{z^0+t\mathbf{b}:|t|=r_1/L(z^0+t_0\mathbf{b})\right\}$ and $r_1\in[r'/4,r'/2]$, for all $j\in\{1,2,\dots,p\}$, by Cauchy's inequality in variable t we obtain

$$\frac{r'^j}{\left(4L(z^0)\right)^j}\left|\frac{\partial^j F(z^0+t_1\mathbf{b})}{\partial\mathbf{b}^j}\right|\leq j!\max\left\{|F(z^0+t\mathbf{b})|:|t-t_1|=\frac{r'}{4L(z^0)}\right\}\leq p!\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{3r'}{4L(z^0)}\right\}$$

that is $g_{z^0}(t_1)\leq p!\max\{1,(4/r')^p\}\max\left\{|F(z^0+t\mathbf{b})|:|t|=\dfrac{3r'}{4L(z^0)}\right\}$. (10)

Inequalities (9) and (10) imply that

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{3r'+1}{4L(z^0)}\right\}\leq P_6\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{3r'}{4L(z^0)}\right\},$$

where $P_6=p!\max\{1,(4/r')^p\}\exp\{3|\mathbf{b}|r'P_5\}$. Hence,

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{6r'+2}{6r'+1}\frac{6r'+1}{8L(z^0)}\right\}\leq P_6\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{6r'}{6r'+1}\frac{6r'+1}{8L(z^0)}\right\}.$$

Denoting $L^*(z)=8L(z)/(6r'+1)$, we obtain

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{6r'+2}{(6r'+1)L^*(z^0)}\right\}\leq P_6\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{6r'}{(6r'+1)L^*(z^0)}\right\}.$$

Therefore, by Theorem A, the function F has a bounded L^*-index in the direction $\mathbf{b}$. And by Theorem 3 from [1] the function F is of the bounded L-index in the direction $\mathbf{b}$ too.

Remark 1. *We require validity of (5) for some r, but nor for all positive r. Thus, Theorem 1 improves the corresponding theorem from [1, 4]. The proposition is new even in the one-dimensional case (see results for the bounded l-index in [6] and bounded index in [8]).*

4. Boundedness of *l*-index of entire solutions of the equation $w'=f(z,w)$

We denote $\mathbf{e}_1=(1,0)$, $\mathbf{e}_2=(0,1)$. $A:=C^2\setminus(G_r^{\mathbf{e}_1}(f)\cup G_r^{\mathbf{e}_2}(f))$

Theorem 2. *Let $l_j\in Q^2_{\mathbf{e}_j}$, $f(z,w)$ be an entire function of bounded l_j-index in the directions $\mathbf{e}_j$ for every $j\in\{1,2\}$. If there exist $C>0$ and $l\in Q$ such that*

$l_1(z,w)+l_2(z,w)|f(z,w)|\le Cl(z)$ for all $(z,w)\in A$, then any entire solution w(z) of (3) has a bounded l-index.

Proof: Differentiating (3) in variable z and using Theorem B we obtain that for all $(z,w)\in\mathbb{C}^2\setminus(G_r^{\mathbf{e}_1}(f)\cup G_r^{\mathbf{e}_2}(f))$

$$|w^{(p+1)}|=\left|\frac{\partial f}{\partial z}+\frac{\partial f}{\partial w}w'\right|\le P(l_1(z,w)|f(z,w)|+l_2(z,w)|f(z,w)|^2)\le PCl(z)|w^{(p)}|.$$

Hence, $l^{-p-1}(z)|w^{(p+1)}(z)|\le PCl^{-p}(z)|w^{(p)}(z)|\le PC\max\{l^{-k}(z)|w^{(k)}(z)|:0\le k\le p\}$. This inequality is similar to (6). Repeating arguments from Theorem 1, we deduce that $w=w(z)$ has a bounded *l*-index. Theorem 2 is proved.

As application of the theorem we consider the differential equation:

$$w^{(p)}=g(z)+\sum_{j=0}^{k}a_jw^j,\ \ a_j\in\mathbb{C}. \tag{11}$$

Corollary 1. *Let $g(z)$ be entire function of bounded l-index, $l^*(z):=l(z)+|g(z)|$, $l^*\in Q$. Then every entire function satisfying* (11) *has a bounded l^*-index.*

5. The linear homogeneous differential equation with fast growing coefficients

As in [10], let φ be a strictly increasing positive unbounded function on $[1,+\infty)$, φ^{-1} be an inverse function to φ, $M(r,f)=\max\{|f(z)|:|z|=r\}$. We define the order of the growth of an entire function $\tilde{\sigma}_\varphi^0[f]=\limsup\limits_{r\to+\infty}\dfrac{\varphi(M(r,f))}{\ln r}$, $\alpha_\varphi=\sup\{\tilde{\sigma}_\varphi^0[g_j]\,|\,j=1,2,\ldots,p\}$ and the function $l_1(z)=\max\limits_{1\le j\le p}(\varphi^{-1}((\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln|z|))^{1/j}$, $|z|>r_1$, where $\varepsilon>0$, r_1 is chosen such that $c=\max\limits_{1\le j\le p}(\phi^{-1}((\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln r_1))^{1/j}\ge 1$. And also we need the greater function $l_0(z)=\phi^{-1}((\alpha_\phi+\varepsilon)\ln|z|)$, $|z|>r_0$, where r_0 is chosen such that $c_0=\varphi^{-1}((\alpha_\varphi+\varepsilon)\ln r_0)\ge 1$. Let K be the class of positive continuously differentiable on $[0,+\infty)$ functions l such that $l'(x)=o(l^2(x))$ as $x\to+\infty$. We need the following proposition of Bordulyak:

Theorem G [9]. *Let $l \in K \cap Q$, and entire functions $g_1, \ldots, g_p$ satisfy the condition $|g_j(z)| \le m_j l^j(|z|)$, $(1 \le j \le p)$ for all z, $|z| \ge R$. If an entire function f is a solution of (4) then f is of the bounded l-index and*

$$\limsup_{r\to+\infty} \ln M(r,f) \Big/ \int_0^r l(t)dt \le \max\left\{1, \sum_{j=1}^p m_j\right\}.$$

Theorem 3. *Let φ be a strictly increasing positive unbounded function on $[1,+\infty)$. If $\alpha_\varphi \in (0,+\infty)$, $l_1 \in Q$, every entire function g_j has a bounded l_1-index ($j \in \{1,\ldots,p\}$) then every entire function satisfying* (4) *has a bounded l_1-index. If, in addition, $l_0 \in Q$, φ is a continuously differentiable function of real variable $t \in [1,+\infty)$ then*

$$\limsup_{r\to+\infty} \ln M(r,f) \Big/ \int_{r_0}^r \phi^{-1}((\alpha_\phi + \varepsilon)\ln t)dt \le N(f,l_0) + 1 \tag{12}$$

for every entire transcendental function f satisfying (4).

Proof: Since $\alpha_\varphi \in (0,+\infty)$, the following inequalities hold $\varphi(M(r,g_j)) < (\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln r$ for arbitrary $\varepsilon > 0$, $r \ge r_0(\varepsilon)$ and $j = 1,\ldots,p$. It means that $M(r,g_j) < \varphi^{-1}((\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln r)$. Denote $r' = \min\{r \ge r_0(\varepsilon) : \varphi^{-1}((\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln r) \ge 1\}$. Hence, for $|z| \ge r'$ one has

$$|g_j(z)| < \varphi^{-1}((\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln|z|) \le (l_1(z))^j,$$

i.e. (5) is valid for $|z| \ge r'$. By Theorems D and 1 entire solutions of (4) have a bounded l_1-index. It is easy to prove that for all $t \in \mathbb{C}$ $l_1(t) \le l_0(t)$. Thus, by Theorem 3 from [1], an entire function f satisfying (4) is of the bounded l_0-index, too. The function φ^{-1} is a strictly increasing and continuously differentiable function of a real variable. Then $(\phi^{-1}(t))' \ge 0$. Furthermore, $(-l_{0'}(t))^+ \equiv 0$ as $t \to +\infty$. Using Theorem F we obtain (12).

Theorem 4. *Let φ be a strictly increasing positive unbounded and continuously differentiable function on $[1,+\infty)$. If $\alpha_\phi \in (0,+\infty)$, $l_1 \in Q$, $t^2\phi'(t)\exp(\varphi(t)/(\alpha_\varphi+\varepsilon)) \to +\infty$ as $t \to +\infty$, then every entire function satisfying (4) has a bounded l_1-index and*

$$\limsup_{r\to+\infty} \ln M(r,f) \Big/ \int_{r_0}^r \varphi^{-1}((\alpha_\varphi + \varepsilon)\ln t)dt \le p.$$

Proof: At first, we prove that $l_1 \in K$. Indeed,

$$\frac{l_1'(t)}{l_1^2(t)} = \frac{(\varphi^{-1}((\alpha_\varphi+\varepsilon)\ln t))'}{(\varphi^{-1}((\alpha_\varphi+\varepsilon)\ln t))^2} = \frac{(\alpha_\varphi+\varepsilon)(\varphi^{-1})'((\alpha_\varphi+\varepsilon)\ln t)}{t(\varphi^{-1}((\alpha_\varphi+\varepsilon)\ln t))^2} = \frac{\alpha_\varphi+\varepsilon}{y^2 e^{\frac{\varphi(y)}{\alpha_\varphi+\varepsilon}}\varphi'(y)} \to 0,$$

where $y=\phi^{-1}((\alpha_\phi+\varepsilon)\ln t)\to+\infty$ as $t\to+\infty$. As above, one has $M(r,g_j)<\varphi^{-1}((\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln r)$. Hence, $|g_j(z)|<\phi^{-1}((\tilde{\sigma}_\varphi^0[g_j]+\varepsilon)\ln|z|)\le(l_1(z))^j$. Thus, g_j and l_1 satisfy conditions of Theorem G with $m_j=1$. Therefore, every entire function satisfying (4) has a bounded l_1-index.

These theorems are a refinement of results of M. Bordulyak, A. Kuzyk and M. Sheremeta [6, 9]. Unlike these authors, we define the specific function l such that entire solutions have a bounded l-index. But the function l depends of the function φ. Below, we will construct functions φ and l_2 for the entire transcendental function f of infinite order.

Theorem 5. *For an arbitrary continuous right differentiable on* $[a,+\infty)$ *function* $l(r)$ *such that* $\limsup\limits_{r\to+\infty} l(r)/r=+\infty$ *there exists a convex on* $[a,+\infty)$ *function* $\Psi(r)$ *with the properties (i)* $l(r)\le\Psi(r)$, $r\ge a$*; (ii)* $l(r)=\Psi(r)$ *for an unbounded from above set of values* $r\ge a$.

Proof: For a given $x\in[a,+\infty)$ we put $\alpha_x(y)=(l(y)-l(x))/(y-x)$ *if* $y>x$ and $\alpha_x(y)=l'_+(x)$ *if* $y=x$. Clearly, the function $\alpha_x(y)$ is continuous on $[x,+\infty)$ and $[l'_+(x),+\infty)$ is fully contained in a range of this function. For every $A>l'_+(x)$ there exists $r>x$ such that $\alpha_x(r)=A$ and $\alpha_x(y)\le A$ for all $y\in[x,r]$. Given the above, it is easy to justify the existence of increasing to $+\infty$ sequence (r_k), for which: 1) $r_0=a$; 2) a sequence (n_k) is increasing to $+\infty$, where $n_k=\alpha_{r_k}(r_{k+1})$ for every $k\ge 0$; 3) $\alpha_{r_k}(y)\le n_k$ for all $y\in[r_k,r_{k+1}]$ and every $k\ge 0$.

Let $k\ge 0$ and $\psi(t)=n_k$ for $t\in[r_k,r_{k+1})$. Clearly, that $\psi(t)$ is a nondecreasing on $[a,+\infty)$ function. Hence, a function $\Psi(r)=l(a)+\int_a^r\psi(t)dt$, $r\ge a$, is convex on $[a,+\infty)$. For this function we have $\Psi(r_0)=\Psi(a)=l(a)=l(r_0)$ and for every $k\ge 1$

$$\Psi(r_k)=l(r_0)+\sum_{j=0}^{k-1}\int_{r_j}^{r_{j+1}}\psi(t)dt=l(r_0)+\sum_{j=0}^{k-1}n_j(r_{j+1}-r_j)=l(r_0)+\sum_{j=0}^{k-1}\alpha_{r_j}(r_{j+1})(r_{j+1}-r_j)=$$
$$=l(r_0)+\sum_{j=0}^{k-1}\frac{l(r_{j+1})-l(r_j)}{r_{j+1}-r_j}(r_{j+1}-r_j)=l(r_0)+\sum_{j=0}^{k-1}(l(r_{j+1})-l(r_j))=l(r_k),$$

i.e. (ii) holds. If $r \in (r_k, r_{k+1})$ for some $k \geq 0$, then we obtain (i):

$$\Psi(r) - \Psi(r_k) = \int_{r_k}^{r} \psi(t)dt = n_k(r - r_k) \geq \alpha_{r_k}(r)(r - r_k) = l(r) - \Psi(r_k).$$

This follows from Theorem 5 that $\Psi'_+(r) \to +\infty$, $r \to +\infty$.

Theorem 6. *For an arbitrary entire transcendental function* f *of infinite order there exists a convex on* $\mathbb{R}$ *function* $\Phi(r)$ *such that 1)* $\ln M_f(t) \leq \Phi(\ln r)$, $r \geq 0$; *2)* $\ln M_f(r) = \Phi(\ln r)$ *for an unbounded from the above set of values* $r > 0$; *3)* $\Phi'_+(r)/\Phi(r) \to +\infty$, $r \to +\infty$.

Proof: We put $l(r) = \ln\ln M_f(e^r)$, $r \geq a$. Since f is of infinite order, it follows $\limsup\limits_{r\to+\infty} l(r)/r = +\infty$. Let $\Psi(r)$ be a function constructed for the function $l(r)$ in Theorem 5. Denote $\Upsilon(r) = \exp(\Psi(r))$, $r \geq a$. Then $\Upsilon'_+(r) = e^{\Psi(r)}\Psi'_+(r)$, $r \geq a$. It means that the function $\Upsilon(r)$ is a convex increasing on half-bounded interval $[b, +\infty)$, where $b \geq a$. We put $\Phi(r) = \Upsilon(r)$ for $r \geq b$ and $\Phi(r) = \Upsilon(b)$ for $r < b$. By Theorem 5 assumptions 1) and 2) hold. We also obtain $\Phi(r) = o(\Phi'_+(r))$, $r \to +\infty$. Therefore, 3) is true.

Let $l_2(z) = \phi^{-1}((\tilde{\sigma}^0_\varphi[f] + \varepsilon)\ln|z|), |z| > r_0$, where r_0 is chosen such that $c_0 = \phi^{-1}((\tilde{\sigma}^0_\varphi[f] + \varepsilon)\ln r_0) \geq 1$.

Theorem 7. *For an arbitrary entire transcendental function* f *of infinite order there exists a strictly increasing positive unbounded and continuously differentiable function* φ *on* $[1, +\infty)$ *with* $\tilde{\sigma}^0_\varphi[f] \in (0, +\infty)$. *And if* $\liminf\limits_{t\to+\infty} t\varphi'(t)\exp(\varphi(t)/s) > 0$, *where* $s = \tilde{\sigma}^0_\varphi[f] + \varepsilon$, *then* $l_2 \in Q$.

Proof: In view of Theorem 6 we choose $\phi(t) := \exp\{\Phi^{-1}(\ln\ln t)\}$, where Φ^{-1} is an inverse function to Φ. Then $l_2(t) = \exp(\exp(\Phi(\ln(s\ln(t)))))$. It is obvious that the function φ is a strictly increasing positive unbounded and $\tilde{\sigma}^0_\varphi[f] = 1$. Besides, φ is a continuously differentiable function except for the points of discontinuity of ϕ. We estimate a logarithmic derivative of l_2:

$$\frac{|l'_2(t)|}{c + |l_2(t)|} = \frac{|(\varphi^{-1}(s\ln t))'|}{c + |\varphi^{-1}(d\ln t)|} = \frac{s|(\varphi^{-1})'(s\ln t)|}{|t|(c + |\varphi^{-1}(d\ln t)|)} = \frac{s}{(c + |y|)e^{\frac{\varphi(y)}{s}}\varphi'(y)} \to 0$$

where $y = \varphi^{-1}(s\ln t) \to +\infty$ as $t \to +\infty$. It implies that $l_2 \in Q$.

6. Conclusions

Note that a concept of the bounded L-index in a direction has a few advantages in the comparison with traditional approaches to study the properties of entire solutions of differential equations. In particular, if an entire solution has a bounded index, then it immediately yields its growth estimates, a uniform in a some sense distribution of its zeros, a certain regular behavior of the solution, etc.

References

[1] Bandura A.I., Skaskiv O.B., Entire functions of bounded L-index in direction, Mat. Stud. 2007, 27, 1, 30-52 (in Ukrainian).

[2] Bandura A.I., Skaskiv O.B., Sufficient sets for boundedness L-index in direction for entire functions, Mat. Stud. 2007, 30, 2, 30-52.

[3] Bandura A.I., Skaskiv O.B., Open problems for entire functions of bounded index in direction, Mat. Stud. 2015, 43, 1, 103-109. dx.doi.org/10.15330/ms.43.1.103-109.

[4] Bandura A., Skaskiv O., Entire functions of several variables of bounded index, Lviv, Chyslo, Publisher I.E. Chyzhykov 2016, 128 p.

[5] Kuzyk A.D., Sheremeta M.M., Entire functions of bounded l-distribution of values, Math. Notes 1986, 39, 1, 3-8.

[6] Kuzyk A.D., Sheremeta M.N., On entire functions, satisfying linear differential equations, Diff. Equations 1990, 26, 10, 1716-1722. (in Russian).

[7] Fricke G.H., Shah S.M., Entire functions satisfying a linear differential equation, Indag. Math. 1975, 37, 39-41.

[8] Fricke G.H., Shah S.M., On bounded value distribution and bounded index Nonlinear Anal. 1978, 2, 4, 423-435.

[9] Bordulyak M.T., On the growth of entire solutions of linear differential equations, Mat. Stud. 2000, 13, 2, 219-223.

[10] Cao T.-B., Complex oscillation of entire solutions of higher-order linear differential equations, Electronic Journal of Differential Equations 2006, 81, 1-8.

[11] Chen Z.X., Yang C.C., Quantitative estimations on the zeros and growth of entire solutions of linear differential equations, Complex Variables 2000, 42, 119-133.

[12] Hamani K., Belaïdi B., Growth of solutions of complex linear differential equations with entire coefficients of finite iterated order, Acta Universitatis Apulensis 2011, 27, 203-216.

[13] Heittokangas J., Korhonen R., Rattyä J., Growth estimates for analytic solutions of complex linear differential equations, Ann. Acad. Sci. Fenn. Math. 2004, 29, 233-246.

[14] Kinnunen L., Linear differential equations with solutions of finite iterated order, Southeast Asian Bull. Math. 1998, 22, 385-405.

[15] Lin J., Tu J., Shi L.Z., Linear differential equations with entire coefficients of $[p;q]$-order in the complex plane, J. Math. Anal. Appl. 2010, 372, 55-67.

[16] Chyzhykov I., Semochko N., Fast growing enitre solutions of linear differential equations, Math. Bull. Shevchenko Sci. Soc. 2016, 13, 68-83.

[17] Sheremeta M., Analytic Functions of Bounded Index, VNTL Publishers, Lviv 1999, 142 p.