EXACT SOLUTION OF FIN PROBLEM WITH LINEAR TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

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Abstract: In this paper, we obtain the general exact solution of a nonlinear fin equation which governs heat transfer in a rectangular fin with linear temperature-dependent thermal conductivity using the partial Noether method. The relationship between the fin efficiency and the thermo-geometric fin parameter is obtained. Additionally, we obtained the relationship among the fin effectiveness, the thermo-geometric fin parameter and the Biot number.

Keywords: exact solution, fin equation, fin efficiency, thermal conductivity

1. Introduction

In this paper, we assume that the rectangular fin subjected to some assumptions such as steady state heat transfer operation with no heat generation, the fin tip is insulated, and the heat transfer is one dimensional. Under these assumptions, the energy balance equation of rectangular fin is given by [1-11]

$$A_c \frac{d}{d\mathcal{X}}\left(K(T)\frac{dT}{d\mathcal{X}}\right) - Ph(T - T_a) = 0, \tag{1}$$

where, T is the fin temperature, $\mathcal{X}$ is the axial distance measured from the fin tip, A_c is the cross-sectional area of the fin, P is the fin perimeter, $k(T)$ is the thermal conductivity of the fin, h is the heat transfer coefficient and T_a is the ambient temperature.

Here, we take the heat transfer coefficient h as a constant and the thermal conductivity $k(T)$ as a linear function of temperature [1-9]

$$k(T) = K_a\big(1 + \tau(T - T_a)\big), \tag{2}$$

where K_a is the thermal conductivity of the fin at the ambient temperature T_a, τ is a constant.

Substituting (2) into (1), we obtain

$$A_c K_a \frac{d}{d\mathcal{X}}\left(\left(1 + \tau(T - T_a)\right)\frac{dT}{d\mathcal{X}}\right) - Ph(T - T_a) = 0. \tag{3}$$

To make Eq. (3) dimensionless; the following transformations are introduced [1-14]

$$u = \frac{T - T_a}{T_b - T_a}, \quad x = \frac{\mathcal{X}}{b}, \quad \beta = \tau(T_b - T_a), \quad \psi^2 = \frac{Phb^2}{K_a A_c}, \tag{4}$$

where, b is the length of fin, T_b is the temperature of the heat source where the fin is attached and the parameter ψ is called thermo geometric fin parameter. Using the transformations (4), Eq. (3) becomes

$$\frac{d}{dx}\left((1 + \beta u)\frac{du}{dx}\right) - \psi^2 u = 0. \tag{5}$$

Equation (5) can be rewritten as

$$u''(x) + \beta u\, u''(x) + \beta\, u'(x)^2 - \psi^2 u(x) = 0. \tag{6}$$

The boundary conditions are:

At the fin tip ($\mathcal{X} = 0$), since the fin tip is insulated, so

$$\frac{dT}{d\mathcal{X}} = 0. \tag{7}$$

Using the transformation (4), Eq. (7) becomes

$$u'(0) = 0. \tag{8}$$

At the fin base ($\mathcal{X} = b$), the fin temperature is the same temperature as the heat source T_b

$$T(b) = T_b. \tag{9}$$

Using the transformation (4), Eq. (9) becomes

$$u(1) = 1. \tag{10}$$

Approximate solutions of Eq. (6) with boundary conditions (8) and (10) are investigated using the Parameterized Perturbation method in [1], by using optimal homotopy asymptotic method in [2], by using the homotopy analysis method in [3, 4], by using the Residue minimization technique in [5], by using the variational iteration method in [6] and by using the decomposition method in [7, 8].

The homotopy analysis method is widely used in investigating many fin problems in [12-14]. In this paper, we will obtain the exact solution of Eq. (6) using the partial Noether method.

The paper will be organized as follows: In section 2, the exact solution of Eq. (6) is obtained using the partial Noether method. In section 3, the fin efficiency will be discussed. In section 4, the fin effectiveness is studied. In section 5, we will discuss the obtained results in this paper.

2. Partial Noether Method

Definition [15, 16]. A Lie operator X of a form

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + (D_x \eta - u' D_x \xi) \frac{\partial}{\partial u'} + \cdots \tag{11}$$

is called a partial Noether operator corresponding to a partial Lagrangian L, if there exists a function B, such that

$$X(L) + L D_x(\xi) = (\eta - u' \xi) \frac{\delta L}{\delta u} + D_x(B), \tag{12}$$

where D_x is the total differentiation with respect to x and $\frac{\delta}{\delta u}$ is called the Euler--Lagrange operator, which are defined as,

$$D_x = \frac{\partial}{\partial x} + u' \frac{\partial}{\partial u} + u'' \frac{\partial}{\partial u'} + \cdots$$

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \left(\frac{\partial}{\partial u'} \right) + \cdots \tag{13}$$

Theorem [15, 16]. If the Lie operator (11) is a partial Noether operator corresponding to a partial Lagrangian L of Eq. (6), then the first integral I of (6) is given by

$$I = B - N(L), \tag{14}$$

which is satisfied by the conservation law

$$D_x(I)|_{(6)} = 0, \tag{15}$$

where, N is a Noether operator which is defined as:

$$N = \xi + (\eta - u' \xi) \frac{\delta}{\delta u'} + \cdots, \tag{16}$$

where,

$$\frac{\delta}{\delta u'} = \frac{\partial}{\partial u'} - D_x\left(\frac{\partial}{\partial u''}\right) + \cdots \tag{17}$$

Consider the partial Lagrangian of Equation (6) [15, 16]

$$L = \frac{(u')^2}{2}, \tag{18}$$

where,

$$\frac{\delta L}{\delta u} = -u'' = \frac{\beta \, (u')^2 - \psi^2 u}{1 + \beta u}. \tag{19}$$

To obtain the partial Noether operator of Eq. (6), we will substitute (18) and (19) into the condition (12) to obtain

$$X\left(\frac{(u')^2}{2}\right) + \frac{(u')^2}{2} D_x(\xi) = (\eta - u'\xi)\left(\frac{\beta \, (u')^2 - \psi^2 u}{1 + \beta u}\right) + D_x(B). \tag{20}$$

Substituting (11) into (20), we obtain the determining equation

$$(D_x\eta - u'D_x\xi)u' + \frac{(u')^2}{2} D_x(\xi) = (\eta - u'\xi)\left(\frac{\beta \, (u')^2 - \psi^2 u}{1 + \beta u}\right) + D_x(B). \tag{21}$$

Let, $\xi = \xi(x, u)$, $\eta = \eta(x, u)$ and $B = B(x, u)$, the determining equation (21) becomes

$$-B_x + \frac{\psi^2 u}{\beta u + 1}\eta + \left(-B_u - \frac{\psi^2 u}{\beta u + 1}\xi + \eta_x\right)u' + \left(\frac{\beta}{\beta u + 1}\xi - \frac{1}{2}\xi_u\right)(u')^3$$
$$+ \left(-\frac{\beta \, \eta}{\beta u + 1} + \eta_u - \frac{1}{2}\xi_x\right)(u')^2 = 0. \tag{22}$$

Equating the coefficients of the derivatives of u with zero, we obtain

$$(u')^3: \qquad\qquad \frac{\beta}{\beta u + 1}\xi - \frac{1}{2}\xi_u = 0, \tag{23}$$

$$(u')^2: \qquad\qquad -\frac{\beta \, \eta}{\beta u + 1} + \eta_u - \frac{1}{2}\xi_x = 0, \tag{24}$$

$$(u'): \qquad\qquad -B_u - \frac{\psi^2 u}{\beta u + 1}\xi + \eta_x = 0, \tag{25}$$

$$(u')^0: \qquad\qquad -B_x + \frac{\psi^2 u}{\beta u + 1}\eta = 0. \tag{26}$$

The solution of system (23)-(26) is given by

$$\xi = 6(1 + \beta u)^2, \quad \eta = 0, \quad B = -u^2(3 + 2\beta u)\psi^2. \tag{27}$$

Substituting (18) into (16), we obtain

$$N(L) = \eta \, u' - \xi \frac{(u')^2}{2}. \tag{28}$$

Substituting (27) and (28) into (14), we obtain

$$I = -u^2(3 + 2\beta u)\psi^2 + 3(1 + \beta u)^2(u')^2. \tag{29}$$

Suppose the first integral $I = c_1$, hence, we obtain

$$-u^2(3 + 2\beta u)\psi^2 + 3(1 + \beta u)^2(u')^2 = c_1, \tag{30}$$

where c_1 is a constant. Using the boundary condition (8), we can determine the constant c_1 as follows

$$-u_0^2(3 + 2\beta u_0)\psi^2 = c_1, \tag{31}$$

where, u_0 is the temperature of fin at the fin tip $(x = 0)$. Substituting (31) into (30), we obtain

$$u_0^2(3 + 2\beta u_0)\psi^2 - u^2(3 + 2\beta u)\psi^2 + 3(1 + \beta u)^2(u')^2 = 0, \tag{32}$$

Let,

$$u = g(x) - \frac{1}{\beta}. $$

Hence, Eq. (32) becomes

$$g'(x)^2 = \frac{\psi^2}{3\beta^4 g(x)^2}\left(-(\beta u_0 + 1)^2(2\beta u_0 - 1) + 2\beta^3 g(x)^3 - 3\beta^2 g(x)^2\right). \tag{33}$$

Integrating Eq. (33), we obtain

$$\int_0^1 dx = \frac{-\sqrt{3}\beta^2}{\psi} \int_{u_0 + \frac{1}{\beta}}^{u + \frac{1}{\beta}} g\left((-(\beta u_0 + 1)^2(2\beta u_0 - 1) + 2\beta^3 g^3 - 3\beta^2 g^2)\right)^{-\frac{1}{2}} dg. \tag{34}$$

Hence, we obtain the following exact implicit solution of Eq. (5)

$$x = -\frac{\sqrt{3}}{\sqrt{-2\gamma}\,\psi}\left(\gamma\,E\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}u_0}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)\right.$$

$$-\gamma\,E\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}u}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)$$

$$+ 4(\beta u_0 + 1)\left(F\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}u_0}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)\right.$$

$$\left.\left.-F\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}u}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)\right)\right), \tag{35}$$

where, E is the incomplete elliptic integral of the second kind, which is defined as [17]

$$E(\phi|m) = \int_0^\phi \sqrt{1 - m\sin^2(\theta)}\, d\theta,$$

F is the incomplete elliptic integral of the first kind, which is defined as [17]

$$F(\phi|m) = \int_0^\phi \frac{1}{\sqrt{1 - m\sin^2(\theta)}}\, d\theta,$$

and $\alpha = \sqrt{3 - 4\beta u_0 - 4\beta^2 u_0^2}$, $\gamma = -3 + \sqrt{3}\alpha - 6\beta u_0$, $\rho = \sqrt{3} - \frac{1}{\alpha}(3 + 2u_0\beta)$. The solution (35) has an unknown parameter namely u_0. This parameter can be easily determined with the help of the boundary condition $u(1) = 1$ as follows:

$$1 = -\frac{\sqrt{3}}{\sqrt{-2\gamma}\,\psi}\left(\gamma\,E\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}u_0}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)\right.$$

$$-\gamma\,E\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)$$

$$+ 4(\beta u_0 + 1)\left(F\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}u_0}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)\right. \tag{36}$$

$$\left.\left.-F\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\sqrt[4]{3}}\sqrt{\rho - \frac{4\beta}{\alpha}}\right)\left|\frac{2\sqrt{3}\alpha}{\gamma}\right.\right)\right)\right).$$

Equation (36) shows the relation between the temperature at fin tip u_0 and the thermo-geometric parameter ψ and β.
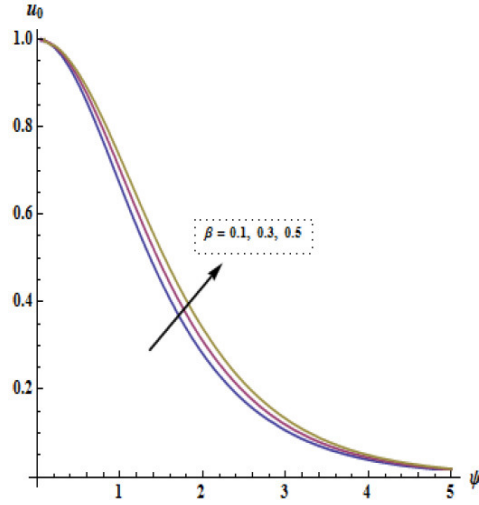

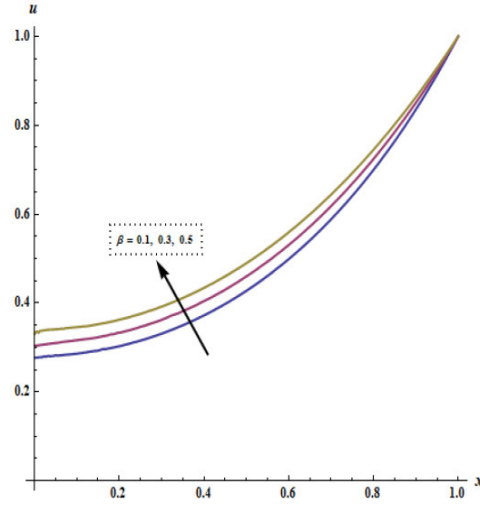Fig. 1. Plot of the relation (36) between ψ and u_0 for various values of β

Fig. 2. Plot of the relation (35) between u and x for various values of β when $\psi = 2$

Figure 1 shows the effect of the thermo-geometric parameter ψ on the fin tip temperature u_0. We find that the fin tip temperature u_0 decreases with increasing ψ. Figure 2 shows the distribution of fin temperature u along the fin. We find that the fin temperature decreases with increasing x.

3. Fin efficiency

The fin efficiency $\boldsymbol{\eta}$ is the ratio of the actual heat transfer rate Q_a from the fin to ideal heat transfer Q_i rate from the fin if the entire fin were at base temperature [3-11]

$$\boldsymbol{\eta} = \frac{Q_a}{Q_i} = \frac{\int_0^b p\, h\, (T - T_a)\, d\mathcal{X}}{pb\, h(T_b - T_a)} = \int_0^1 u\, dx. \tag{37}$$

Using Eq. (5), Eq. (37) becomes

$$\boldsymbol{\eta} = \frac{1}{\psi^2} \int_0^1 \frac{d}{dx}\left((1 + \beta u)\frac{du}{dx}\right) dx = \frac{1}{\psi^2}\left((1 + \beta u(1))u'(1) - (1 + \beta u(0))u'(0)\right), \tag{38}$$

Using the boundary conditions (8) and (10), Eq. (38) becomes

$$\boldsymbol{\eta} = \frac{(1 + \beta)}{\psi^2} u'(1), \tag{39}$$

From (32), when $x = 1$, we obtain

$$u'(1) = \frac{\psi}{\sqrt{3}(1 + \beta)} \sqrt{(3 + 2\beta) - u_0^2(3 + 2\beta u_0)}. \tag{40}$$

Substituting (40) into (39), we obtain

$$\boldsymbol{\eta} = \frac{1}{\sqrt{3}\psi} \sqrt{(3 + 2\beta) - u_0^2(3 + 2\beta u_0)}. \tag{41}$$

Using the relations (41) and (36), we can plot the relation between the efficiency $\boldsymbol{\eta}$ and the thermo-geometric fin parameter ψ (see Figure 3).

3. Fin effectiveness

Fin effectiveness ϵ is the ratio of heat transferred from the fin area Q_a to the heat which would be transferred if entire fin area was at base temperature Q_b [11]

$$\epsilon = \frac{Q_a}{Q_b} = \frac{\int_0^b p\, h\, (T - T_a)\, d\mathcal{X}}{A_c h (T_b - T_a)} = G \int_0^1 u\, dx. \tag{42}$$

where, $G = \dfrac{Pb}{A_c}$ is a parameter which depends on the fin geometry.
From (37), we find

$$\epsilon = G\, \boldsymbol{\eta}. \tag{43}$$

Substituting (41) into (43), we obtain

$$\epsilon = \frac{G}{\sqrt{3}\psi} \sqrt{(3 + 2\beta) - u_0^2(3 + 2\beta u_0)}. \tag{44}$$

The parameter ψ can be rewritten in the form

$$\psi^2 = Bi\, G. \tag{45}$$

where, $Bi = \dfrac{hb}{K_a}$ is the Biot number.
Substituting (45) into (44), we obtain

$$\epsilon = \sqrt{\frac{G}{3Bi} \left((3 + 2\beta) - u_0^2(3 + 2\beta u_0)\right)}. \tag{46}$$

Using the relations (45), (46) and (36), we can plot the relation between the fin effectiveness ϵ and the parameters G and Bi (Figures 4 and 5).
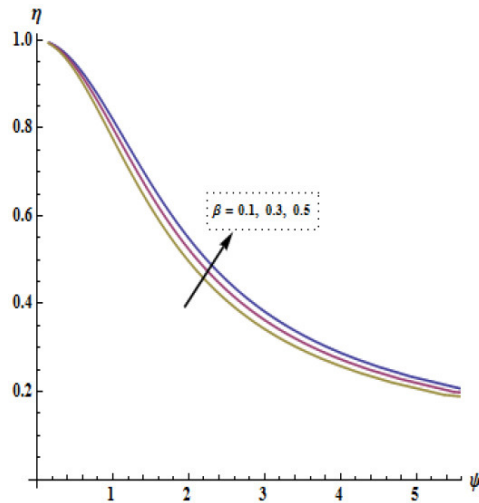
Fig. 3. Plot the relation between the efficiency η and the thermo-geometric parameter ψ for various values of β
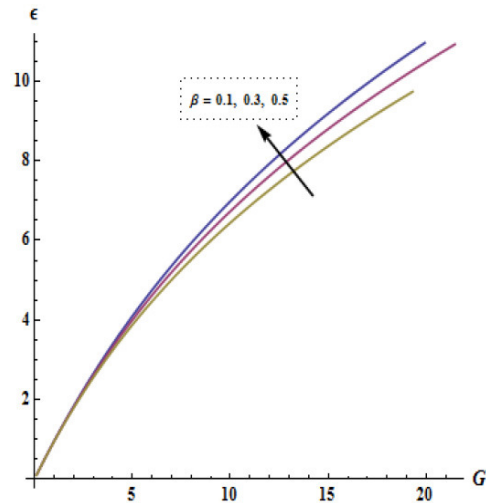
Fig. 4. Plot the relation between the effectiveness of fin ϵ and G when $Bi = 0.2$ for various values of β

Figure 3 shows the effect of the thermo-geometric parameter ψ on fin efficiency η. We find that the fin efficiency η decreases with increasing ψ. Figure 4 shows the effect of the parameter G on fin effectiveness ϵ. We find that fin effectiveness ϵ increases with increasing G.
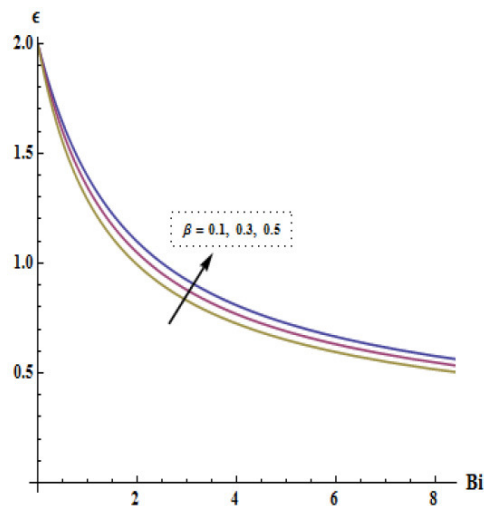

Fig. 5. Plot the relation between the fin effectiveness ϵ and Bi when $G = 2$ for various values of β

Figure 5 shows the effect of the Biot number on fin effectiveness ϵ. We find that fin effectiveness ϵ decreases with increasing Bi.

5. Discussions and concluding remarks

In this paper, we obtain the general exact solution (35) of the fin equation (5) which is subjected to the boundary conditions (8) and (10). The solution is valid for all values of the thermo-geometric fin parameters ψ and β. We observe in Figure 1 that the fin tip temperature u_0 decreases with an increase in the thermo-geometric parameter ψ. Figure 2 shows that the fin temperature u increases with an increasing x (in other words, the temperature increases when approaching a heat source). The relation between the fin efficiency η and the parameters ψ and β is obtained. Figure 3 shows that the fin efficiency η decreases with increasing the thermo-geometric parameter ψ. The relation between the fin effectiveness ϵ and the parameters, β and Biot number Bi is obtained. Figures 4 and 5 show that the fin effectiveness ϵ increases when increasing G and decreases when increasing the Biot number Bi.

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