Journal of Applied Mathematics and Computational Mechanics 2016, 15(3), 47-56
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2016.3.05 e-ISSN 2353-0588

# REPRESENTATIONS OF (*D*,*O*)-SPECIES AND FLAT MIXED MATRIX PROBLEMS

***Nadiya Gubareni***

*Institute of Mathematics, Czestochowa University of Technology*
*Częstochowa, Poland*
*nadiya.gubareni@yahoo.com*

**Abstract.** The problem of describing representations of $(D,O)$-species is reduced to flat mixed matrix problems over discrete valuation rings and their common skew field of fractions.

***Keywords:*** *O-species, (D,O)-species, representations of (D,O)-species, (D,O)-species of bounded representation type, flat mixed matrix problem, discrete valuation ring*

## 1. Introduction

We continue the study of $(D,O)$-species that was started in [1]. These species generalize the notion of species introduced by Gabriel [2] and are the special kind of species considered in [3].

Let $\{O_i\}$ be a family of discrete valuation rings (not necessarily commutative) with a common skew field of fractions $D$. Consider a $(D,O)$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$, where $F_i = H_{n_i}(O_i)$ for $i = 1, 2, ..., k$, and $F_j = D$ for $j = k+1, ..., n$, moreover ${}_iM_j$ is an $(\widetilde{F}_i, \widetilde{F}_i)$-bimodule that is finite dimensional both as the left $D$-vector space and as the right $D$-vector space, where $\widetilde{F}_i$ is a classical ring of fractions of $F_i$ for $i = 1, 2, ..., n$.

A $(D,O)$-species $\Omega$ is called **weak** if $F_i = O_i$ for all $i = 1, 2, ..., k$, and moreover, ${}_iM_j = 0$ if $F_j = O_j$, and ${}_iM_j = {}_jM_i = 0$ for $i,j \in I$ and $i \neq j$.

For $(D,O)$-species the representations of $O$-species were defined in [1]. A representation $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ of a weak $(D,O)$-species $\Omega = \{F_i, {}_iM_j\}_{i,j\in I}$ is a family of right $F_i$-modules $M_i$ ($i = 1, 2, ..., k$), a set of right $D$-vector spaces $V_r$ ($r = k+1, k+1, ..., n$) and $D$-linear maps:

$$ {}_j\varphi_i : M_i \otimes_{F_i} {}_iM_j \to V_j $$

for each $i = 1, 2, ..., k$; $j = k+1, k+2, ..., n$; and

$$_j\psi_r : V_r \otimes_D {}_rM_j \to V_j$$

for each $r, j = k+1, k+2, ..., n$.

A representation $V$ is said to be finite dimensional if all $M_i$ are finitely generated $F_i$-modules and all $V_r$ are finite dimensional $D$-vector spaces. A $(D,O)$-species is of bounded representation type if the dimensions (see (3.13) in [1]) of its indecomposable finite dimensional representations have an upper bound.

In this paper, we show that the description of representations of $(D,O)$-species can be reduced to some flat mixed matrix problems over discrete valuation rings and their common skew field of fractions. The definition of such matrix problems is given in Section 2. These matrix problems are some sort of generalization of a flat matrix problem considered by Zavadskii and Revitskaya [4]. Earlier such matrix problems were considered by Gubareni [5, 6], and Zavadskii and Kirichenko [7, 8]. Some examples of such flat matrix problems were also considered in [9]. The reduction of the problem of description of $(D,O)$-species of bounded representation type to some flat mixed matrix problems is given in Section 3.

With each weak $(D,O)$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ we can associate a $D$-species $\widetilde{\Omega} = (\widetilde{F}_i, {}_iM_j)_{i,j\in I}$, where $\widetilde{F}_i = D$. In Section 4, we prove that if $\Omega$ is a simply connected weak $(D,O)$-species of bounded representation type, then $\widetilde{\Omega}$ is a $D$-species of finite representation type.

## 2. Flat mixed matrix problems

Let $O$ be a discrete valuation ring (DVR) with a classical division ring of fractions $D$. By left $O$-elementary transformations of rows of a matrix $\mathbf{T}$ with entries in $D$ we mean transformations of two types:
a) multiplying a row on the left by an invertible element of $O$;
b) adding a row multiplied on the left by an element of $O$ to another row.

In a similar way we can define left $D$-elementary transformations of rows and, by symmetry, right $O$-elementary and right $D$-elementary transformations of columns.

Elementary transformations (a) and (b) can be given by invertible elementary matrices. The automorphism of a finitely generated module $P$ corresponding to an elementary transformation is an elementary automorphism. Multiplications on the left (right) side of a matrix $\mathbf{T}$ by elementary matrices correspond to elementary row (column) transformations.

By [10, Proposition 13.1.3], any invertible matrix $\mathbf{B}$ over a local ring $O$ can be reduced by $O$-elementary row (column) transformations on $\mathbf{B}$ to the identity matrix. By [10, Corollary 13.1.4], the matrix $\mathbf{B}$ can be decomposed into a product of elementary matrices. Moreover, by [10, Theorem 13.1.6] any automorphism of a finitely generated projective module $P$ over a semiperfect ring $A$ can be decomposed into a product of elementary automorphisms.

Let $\Delta = \{O_i\}_{\{i=1,\ldots,k\}}$ be a family of discrete valuation rings $O_i$ with a common skew field of fractions $D$. We define the general flat matrix problem over $\Delta$ and $D$ in the following way.

Let

| | | | | |
|---|---|---|---|---|
| $\mathbf{T}_{11}$ | … | $\mathbf{T}_{1j}$ | … | $\mathbf{T}_{1m}$ |
| … | … | … | … | … |
| $\mathbf{T}_{i1}$ | … | $\mathbf{T}_{ij}$ | … | $\mathbf{T}_{1m}$ |
| … | … | … | … | … |
| $\mathbf{T}_{n1}$ | … | $\mathbf{T}_{nj}$ | … | $\mathbf{T}_{nm}$ |

be a block rectangular matrix $\mathbf{T}$ with entries in $D$ partitioned into $n$ horizontal strips $\mathbf{T}_1, \ldots, \mathbf{T}_n$ and $m$ vertical strips $\mathbf{T}^1, \ldots, \mathbf{T}^m$ so that each block $\mathbf{T}_{ij}$ is the intersection of the $j$-th vertical strip and the $i$-th horizontal strip; some of these blocks may be empty.

Assume that the ring $F_{i_s} \in \Delta \cup D$ corresponds to the $i$-th horizontal strip $\mathbf{T}_i$ and the ring $F_{j_t} \in \Delta \cup D$ corresponds to the $j$-th vertical strip $\mathbf{T}^j$.

The following transformations with the matrix $\mathbf{T}$ are admissible:

1. Left $F_{i_s}$-elementary transformations of rows within the strip $T_i$.
2. Right $F_{j_t}$-elementary transformations of rows within the strip $T^j$.
3. Additions of rows in the strip $T_j$ multiplied on the left by elements of $F_r \in \Delta \cup D$ to rows in the strip $T_i$.
4. Additions of columns in the strip $T^i$ multiplied on the right by elements of $F_p \in \Delta \cup D$ to columns in the strip $T^j$.

Indecomposable matrices and equivalent matrices are defined in a natural way.

A flat matrix problem is said to be of **finite type** if the number non-equivalent indecomposable matrices is finite.

**Definition 2.1.** The vector

$$\mathrm{d} = \mathrm{d}(\mathbf{T}) = (d_1, d_2, \ldots, d_n; d^1, d^2, \ldots, d^m), \tag{2.2}$$

where $d_i$ is the number of rows of the $i$-th horizontal strip of $\mathbf{T}$ for $i = 1, \ldots, n$ and $d^j$ is the number of columns of the $j$-th vertical strip of $\mathbf{T}$ for $j = 1, \ldots, m$, is called the **dimension vector** of the partition matrix $\mathbf{T}$. Also set

$$\dim(\mathbf{T}) = \sum_{i=1}^{n} d_i + \sum_{j=1}^{m} d^j \tag{2.3}$$

**Definition 2.4.**
A flat matrix problem is said to be of **bounded representation type** if there is a constant $C$ such that $\dim(\mathbf{X}) < C$ for all indecomposable matrices $\mathbf{X}$. Otherwise it is of **unbounded representation type**.

## 3. The main matrix problem

Let $\Omega = (F_i, {}_iM_j)_{i,j\in I}$, where $F_i = O_i$ for $i = 1,2, \ldots, k$ and $F_j = D$ for $j = k+1, \ldots, n$, be a weak $(D,O)$-species of bounded representation type.

Suppose that $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ is an indecomposable finite dimensional representation of $\Omega$. Then $M_i$ is a finitely generated $F_i$-module for $i = 1,2, \ldots, k$ and $V_r$ is a finite dimensional $D$-vector space for $r = k+1, \ldots, n$. Since $F_i = O_i$ is a discrete valuation ring, by [3, Proposition 5.4.18], any $O_i$-module $M_i$ is torsion-free and faithful. Therefore any indecomposable representation of $\Omega$ has the following form:

$$V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r) \tag{3.1}$$

where $M_i$ is a free $F_i$-module.

Consider the category $R(\Omega)$ whose objects are representations $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$, and a morphism from an object $V$ to an object $V' = (M_i', V_r', {}_j\varphi_i', {}_j\psi_r')$ is a set of homomorphisms $(\alpha_i, \beta_r)$, in which $\alpha_i : M_i \to M_i'$ is a homomorphism of $F_i$-modules, $\beta_r : V_r \to V_r'$ is a homomorphism of $D$-vector spaces ($r = k+1, \ldots, n$), and the following equalities hold:

$${}_j\varphi_i'(\alpha_i \otimes 1) = \beta_j \cdot {}_j\varphi_i \tag{3.2}$$

$${}_j\psi_r'(\beta_r \otimes 1) = \beta_j \cdot {}_j\psi_r \tag{3.3}$$

Let $V$ be an indecomposable finite dimensional representation of the $(D,O)$-species $\Omega$. Thus, each $M_i$ is a finitely generated free $O_i$-module with basis $\omega_1^{(i)}, \ldots, \omega_{m_i}^{(i)}$ ($i = 1,2, \ldots, k$); and $V_r$ is a finite dimensional $D$-space with basis $\tau_1^{(r)}, \ldots, \tau_{k_r}^{(r)}$ ($r = k+1, \ldots, n$).

Suppose

$${}_j\varphi_i(\omega_s^{(i)} \otimes 1) = \sum_{u=1}^{k_j} \tau_u^{(j)} b_{us}^{(ij)} \tag{3.4}$$

$${}_j\psi_r(\tau_v^{(r)} \otimes 1) = \sum_{u=1}^{k_j} \tau_u^{(j)} a_{uv}^{(ij)} \tag{3.5}$$

Then the matrices $\mathbf{A}_{ij} = (a_{uv}^{(ij)})$, $\mathbf{B}_{ij} = (b_{us}^{(ij)})$ define the representation $V$ uniquely up to equivalence.

Let $\mathbf{U}_i \in M_{m_i}(F_i)$ be the matrix corresponding to the homomorphism $\alpha_i$, and let $\mathbf{W}_i \in M_{k_i}(D)$ be the matrices corresponding to the homomorphisms $\beta_i$, $i \in \mathrm{I}$. If $\mathbf{A}'_{ij}$, $\mathbf{B}'_{ij}$ are the matrices corresponding to a representation $V'$ then the equalities (3.2) and (3.3) have the following matrix form:

$$\mathbf{W}_i\,\mathbf{B}_{ij} = \mathbf{B}'_{ij}\,\mathbf{U}_j\ (i=1, \ldots, k; j = k+1, \ldots, n) \tag{3.6}$$

$$\mathbf{W}_j\,\mathbf{A}_{jr} = \mathbf{A}'_{jr}\,\mathbf{W}_r\ (j, r = k+1, \ldots, n) \tag{3.7}$$

If representations $V$ and $V'$ are equivalent, then $\alpha_i$, $\beta_r$ are isomorphisms. Therefore, the matrices $\mathbf{U}_i$ and $\mathbf{W}_r$ are invertible and the equalities (3.2) and (3.3) are equivalent to the following equalities:

$$\mathbf{W}_i\,\mathbf{B}_{ij}\,\mathbf{U}_j^{-1} = \mathbf{B}'_{ij}\ (i = 1, \ldots, k; j = k+1, \ldots, n) \tag{3.8}$$

$$\mathbf{W}_j\,\mathbf{A}_{jr}\,\mathbf{W}_r^{-1} = \mathbf{A}'_{jr}\ (j, r = k+1, \ldots, n) \tag{3.9}$$

Thus we obtain the following matrix problem for description of indecomposable finite dimensional representations of a (*D*,*O*)-species $\Omega$.

***Main mixed matrix problem***

Let $\Delta = \{O_i\}_{i=1,2,\ldots,k}$ be a family of discrete valuation rings $O_i$ with a common skew field of fractions $D$.

Let $\mathbf{T}$ be a block matrix with entries in $D$ partitioned into $n$ horizontal strips $\{\mathbf{T}_i\}_{\{i=1,\ldots,n\}}$ and $m$ vertical strips $\{\mathbf{T}^j\}_{\{j=1,\ldots,m\}}$ so that each block $\mathbf{T}_{ij}$ is the intersection of $j$-th vertical strip and $i$-th horizontal strip, some of these matrices may be empty.

The following transformations with the matrix $\mathbf{T}$ are admissible:

1. Left $F_{i_s}$-elementary transformations of rows within the strip $\mathrm{T}_i$, where $F_{i_s} \in \Delta \cup D$.
2. Right $F_{j_t}$-elementary transformations of rows within the strip $\mathrm{T}^j$, where $F_{j_t} \in \Delta \cup D$.

The admissible transformations with the matrix $\mathbf{T}$ can be given in the form $\mathbf{T} \to \mathbf{XTY}$, where $\mathbf{X} = \mathrm{diag}\,(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $\mathbf{Y} = \mathrm{diag}\,(\mathbf{Y}_1, \ldots, \mathbf{Y}_m)$, and all $\mathbf{X}_i$ and $\mathbf{Y}_j$ are square invertible matrices. Moreover, $\mathbf{X}_i \subset M_{m_i}(F_{i_s})$, and $\mathbf{Y}_j \subset M_{k_j}(F_{j_t})$, where $F_{i_s}, F_{j_t} \in \Delta \cup D$.

Clearly, the matrix **T** is indecomposable if and only if the corresponding representation of $\Omega$ is indecomposable. It is easy to prove the following statement.

**Lemma 3.10.** *A* $(D,O)$*-species* $\Omega$ *is of bounded representation type if and only if the corresponding main matrix problem is of bounded representation type.*

## 4. Weak $(D,O)$-species of bounded representation type

Let $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ be $O$-species. The **quiver** $\Gamma(\Omega)$ of an $O$-species $\Omega$ is defined as the directed graph whose vertices are $1, \ldots, n$, and there is an arrow from the vertex $i$ to the vertex j if and only if ${}_iM_j \neq 0$.

An $O$-species $\Omega$ is called **acyclic** if its quiver has no oriented cycles, i.e. the indices can be chosen so that ${}_iM_i = 0$ for all $i$, and ${}_iM_j = 0$ for $j \le i$.

A vertex $i \in I$ is called marked if $F_i = H_{n_i}(O_i)$. Let $I_1 = \{1, 2, ..., k\}$ be the set of marked vertices of an $O$-species $\Omega$. A marked vertex $i \in I_1$ is called **minimal** if ${}_iM_j = {}_jM_i = 0$ for all $j \in I_1$. An O-species $\Omega$ is called **min-marked** if all its marked vertices are minimal.

An $O$-species $\Omega$ is **simply connected** if the underlying graph of $\Gamma(\Omega)$ is a tree.

A $(D,O)$-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ is said to be weak if $\Omega$ is min-marked and all $F_i$, are $O_i$ or $D$.

For each $O$-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ in [1], the tensor algebra $\mathrm{T}(\Omega) = \bigoplus_{i=0}^{\infty} T_i$, where $T_0 = \prod_{i=1}^{n} F_i = B$, $T_{i+1} = T_i \otimes_B M$ and $M = \bigoplus_{i,j=1}^{\infty} {}_iM_j$, was constructed.

**Lemma 4.1.** *Let* $\Omega = (F_i, {}_iM_j)_{i,j \in I}$*, where all* $F_i = D$*, be a simply connected D-species of finite representation type. Then the tensor algebra* $\mathrm{T}(\Omega)$ *is a hereditary Artinian semidistributive ring.*

*Proof.* Since $\Omega$ is a simply connected species, the tensor algebra $\mathrm{T}(\Omega)$ is Morita equivalent to the algebra

$$A = \begin{pmatrix} D & & A_{ij} \\ & \ddots & \\ 0 & & D \end{pmatrix}$$

where

$$A_{ij} = \bigoplus_{i = i_0 < i_1 < \ldots < i_k = j} {}_{i_0}M_{i_1} \otimes_{i_1} M_{i_2} \otimes \cdots \otimes_{i_{k-1}} M_{i_k} \tag{4.2}$$

Since all ${}_iM_j$ are finitely dimensional right and left $D$-spaces, $A$ is an Artinian ring. From [11, Corollary 2.2.13] it follows that $A$ is a hereditary ring.

Note that the ring

$$\begin{pmatrix} D & V_{12} \\ 0 & D \end{pmatrix} \tag{4.3}$$

where $V_{12}$ is a $(D, D)$-bimodule, is of finite representation type if and only if $V_{12}$ has dimension 1 both as right and as left $D$-vector space. Since $\Omega$ is a $D$-species of finite representation type, the tensor algebra $T(\Omega)$ is of finite representation type as well, and so it does not contain a minor that is isomorphic to the ring (4.3). Therefore, $A$ is a semidistributive ring.

Besides a weak $(D,O)$-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ we can also consider a $D$-species $\widetilde{\Omega} = (\widetilde{F}_i, {}_iM_j)_{i,j\in I}$, where $\widetilde{F}_i = D$, since each ${}_iM_j$ is an $(\widetilde{F}_i, \widetilde{F}_j)$-bimodule. Let $T(\widetilde{\Omega})$ be a tensor algebra of $D$-species $\widetilde{\Omega}$. Since $T(\widetilde{\Omega})$ is an Artinian ring, by [12, 13] it is of bounded representation type if and only if it is of finite representation type.

**Proposition 4.4.** *If $\Omega$ is a weak simply connected $(D,O)$-species of bounded representation type, then $\widetilde{\Omega}$ is a D-species of finite representation type.*

*Proof.* Let $\Omega$ be a weak simply connected $(D,O)$-species with set of marked vertices $J = \{1,2, ..., k\}$. Then the tensor algebra $A = T(\Omega)$ is a basic primely triangular ring whose two-sided Peirce decomposition has the following form

$$A = T(\Omega) = \begin{pmatrix} O_1 & \cdots & 0 & U_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & O_k & U_k \\ 0 & \cdots & 0 & T \end{pmatrix} \tag{4.5}$$

where each $U_i$ is a $(D,T)$-bimodule. Moreover, the ring $T$ is the tensor algebra of a species $\Omega_1 = (F_i, {}_iM_j)_{i,j \in I\setminus J}$, where $F_i = D$ for all $i \in I\setminus J$.

Since $\Omega$ is a $(D,O)$-species of bounded representation type, then the tensor algebra $T(\Omega)$ is also is of bounded representation type by [1, Corollary 3.15]. Then by [1, Corollary 3.16], $T$ is also of bounded representation type. Since $\Omega_1$ is a $D$-species, $T$ is an Artinian ring and so it is of finite representation type. Since $\Omega$ is simply connected, $\Omega_1$ is also simply connected. By Lemma 4.1, $T$ is an Artinian hereditary semidistributive ring.

Let $\widetilde{A}$ be a right classical ring of fractions of $A$. We will use the following notation: if $M$ is a right $A$-module, then $M' = M \otimes_A \widetilde{A}$; and if $M$ is a right $\widetilde{A}$-module, then $M'$ is the module $M$ considered as an $A$-module. The length of a composition series of a right $\widetilde{A}$-module $X$ is denoted by $l(X)$.

Let us prove that for any right $\widetilde{A}$-module $M$ there is a right $\widetilde{A}$-module $X$ such that $M'' = M \oplus X$.

We have

$$M'' = M'' \otimes_A \widetilde{A} = (M \otimes_A \widetilde{A}) \otimes_A \widetilde{A}.$$

Taking into account (4.5), we have that $M = \bigoplus_{i=1}^{k} M_i \oplus M_0$, where $M_i$ is an $O_i$-module and $M_0$ is a $T$-module. Then

$$M \otimes_A \widetilde{A} = \left( \bigoplus_{i=1}^{k} M_i \oplus M_0 \right) \otimes_A \widetilde{A} = \bigoplus_{i=1}^{k} (M_i \otimes_{O_i} D) \oplus M_0$$

$$M'' = (M \otimes_A \widetilde{A}) \otimes_A \widetilde{A} = \bigoplus_{i=1}^{k} M_i \otimes_{O_i} (D \otimes_{O_i} D) \oplus M_0.$$

By [14, Lemma 2], there is an injective torsion-free $O_i$-module for each $i = 1, \ldots, k$. Therefore, the mapping $D \to D \otimes_{O_i} D$ with $d \mapsto 1 \otimes d$ for each $d \in D$ is a monomorphism, i.e. exact sequences of $O_i$-modules exist:

$$0 \to D \to D \otimes_{O_i} D \to \mathrm{Coker}\,(\varphi_i) \to 0$$

Since $D$ is injective, these sequences split, i.e. $D \otimes_{O_i} D = D \oplus Y_i$ for $i = 1, \ldots, k$. Therefore,

$$M'' = \bigoplus_{i=1}^{k} M_i \otimes_{O_i} (D \oplus Y_i) \oplus M_0 = \bigoplus_{i=1}^{k} ((M_i \otimes_{O_i} D) \oplus (M_i \otimes_{O_i} Y_i)) \oplus M_0 =$$
$$= M \oplus X.$$

Now suppose that the ring $A$ is of bounded representation type and the ring $\widetilde{A}$ is of infinite representation type. Then for any $N > 0$ there is an indecomposable finitely generated $\widetilde{A}$-module $M$ such that $l(M) > N$.

Consider the $A$-module $M'$. It is finitely generated and, by [15, Proposition 1], it decomposes into a direct sum of finitely generated indecomposable $A$-modules:

$$M' = N_1 \oplus \ldots \oplus N_t.$$

Then

$$M'' = N_1' \oplus \ldots \oplus N_t'$$

Since $M'' = M \oplus X$, and $M''$ is a finitely generated module over an Artinian ring $\widetilde{A}$, it follows from the uniqueness of the decomposition that there is a number $i$ such that $M$ is a direct summand of $N_i'$, i.e. there is an $\widetilde{A}$-module $P$ such that $N_i' = M \oplus P$. We have the chain of inequalities

$$\mu_A(N_i) = \mu_{\widetilde{A}}(N_i') \geq l(N_i') = l(M) + l(P) \geq l(M) > \mathrm{N},$$

which contradicts the assumption that $A$ is of bounded representation type.

## 5. Conclusions

The problem of describing representations of $(D,O)$-species has been reduced to some flat matrix problems over discrete valuation rings with common skew field of fractions. The main matrix problem for description of $(D,O)$-species of bounded representation type is given. We establish the connection of $(D,O)$-species of bounded representation type with $D$-species of finite representation type. We prove that if $\Omega$ is a weak simply connected $(D,O)$-species of bounded representation type, then the corresponding $D$-species $\widetilde{\Omega}$ is of finite representation type.

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