Journal of Applied Mathematics and Computational Mechanics 2014, 13(4), 103-108

BUCKLING ANALYSIS OF AXIALLY FUNCTIONALLY GRADED BEAMS

Jowita Rychlewska

Institute of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
jowita.rychlewska@im.pcz.pl

Abstract. In this paper critical buckling loads for axially functionally graded (FG) beams are studied. It is assumed that material properties of the beam vary exponentially through the axial direction. Solutions are derived for three types of boundary conditions: a beam that is clamped at both ends, pinned at both ends and a beam that is clamped at one end and pinned at the other.

Keywords: *functionally graded beams, buckling load*

Introduction

Functionally graded (FG) beams are composed of two or more materials and are characterized by a continuous variance in their material properties in the preferred direction. It is well known that beams are structural elements which carry compressive load. When compressive load crosses a critical value, elastic beam deviates from an original equilibrium state and buckling occurs. A list of papers on buckling aspects of the homogeneous structures is very extensive. For example Kukla and Skalmierski [1] presented the solution to the problem of vibrations of an Euler-Bernoulli beam, which is loaded by an axial force varying along the length of the beam. Exact mathematical solutions for buckling of structural members for various cases of columns, beams, arches, rings, plates and shells are shown in Wang et al. [2]. The buckling analysis of the functionally graded beams is the current field of research. Singh and Li [3] surveyed the stability of axially FG tapered beams by modelling a non-uniform beam as a set of uniform segments and solving a transcendental equation to compute the critical buckling load. The buckling behavior of axially non-uniform elastically restrained beams was studied by Huang and Luo [4]. By expanding the mode shapes as a power series, they transformed the governing differential equations with variable coefficients to a system of algebraic equations. The free vibration and stability of axially functionally graded tapered Euler-Bernoulli were analyzed by Shahba and Rajasekaran [5]. The solution to the problem was obtained by using a differential transform element method and differential quadrature element method of lowest order. The finite element approach to the free vibration and stability analysis of axially functionally graded tapered Timoshenko beams was applied by Shahba et al. [6].

In the present study the stability analysis of axially graded beams with a distributed axial load is made. It is assumed that the changes of material properties as well as the axial load of the beam have an exponential form. The obtained analytical solutions of the buckling analysis for clamped-clamped, pinned-pinned and clamped-pinned beams are used for numerical computations.

1. Formulation of the problem

Consider an axially graded and non-uniform beam of length L (along the x direction) having moment of inertia $I(x)$ and modulus of elasticity $E(x)$, which is loaded by an axial force $P(x)$ varying along the length of the beam. According to the Euler-Bernoulli beam theory, the differential equation that governs the transverse displacement w is given by

$$\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2w}{dx^2}\right]+\frac{d}{dx}\left[P(x)\frac{dw}{dx}\right]=0, \quad 0<x<L, \tag{1}$$

In this paper it is assumed that

$$E(x)I(x)=D_0e^{2\beta\frac{x}{L}} \quad P(x)=P_0e^{2\beta\frac{x}{L}}, \quad 0<x<L, \tag{2}$$

where D_0 is a reference value of EI at $x=0$, P_0 is a reference value of P at $x=0$ and β is the dimensionless parameter. Substituting (2) into (1) and introducing non-dimensional variables

$$\xi=\frac{x}{L}, \quad \lambda=\frac{P_0L^3}{D_0} \tag{3}$$

we can change the governing equation (1) into

$$\frac{d^2}{d\xi^2}\left[e^{2\beta\xi}\frac{d^2w}{d\xi^2}\right]+\lambda\frac{d}{d\xi}\left[e^{2\beta\xi}\frac{dw}{d\xi}\right]=0, \quad 0<\xi<1 \tag{4}$$

After some transformations equation (4) can be written in the form

$$\frac{d^4w}{d\xi^4}+4\beta\frac{d^3w}{d\xi^3}+\left(\lambda+4\beta^2\right)\frac{d^2w}{d\xi^2}+2\beta\lambda\frac{dw}{d\xi}=0, \quad 0<\xi<1 \tag{5}$$

Under assumption $\beta^2 < \lambda$, the general solution of equation (5) has the following form

$$w(\xi) = C_1 + C_2 e^{-2\beta\xi} + C_3 e^{-\beta\xi} \cos\delta\xi + C_4 e^{-\beta\xi} \sin\delta\xi, \quad 0 < \xi < 1, \tag{6}$$

where $\delta = \sqrt{\lambda - \beta^2}$, $C_i \in R$, $i = 1,...,4$. In order to determine critical buckling loads of axially functionally graded beams solution (6) has to be applied to certain boundary conditions.

2. Solution of the problem

In this section we will consider three types of boundary conditions:

– clamped-clamped beams (C-C)

$$\begin{aligned} w(0) = 0 \quad \frac{dw}{d\xi}(0) = 0 \\ w(1) = 0 \quad \frac{dw}{d\xi}(1) = 0 \end{aligned} \tag{7}$$

– pinned-pinned beams (P-P)

$$\begin{aligned} w(0) = 0 \quad \frac{d^2w}{d\xi^2}(0) = 0 \\ w(1) = 0 \quad \frac{d^2w}{d\xi^2}(1) = 0 \end{aligned} \tag{8}$$

– clamped-pinned beams (C-P)

$$\begin{aligned} w(0) = 0 \quad \frac{dw}{d\xi}(0) = 0 \\ w(1) = 0 \quad \frac{d^2w}{d\xi^2}(1) = 0 \end{aligned} \tag{9}$$

By applying one of the system of boundary conditions (7)-(9) to equation (6), we have the homogeneous system of four linear equations with respect to the unknown $C_i, i = 1,...,4$. This system of equations can be written in the matrix form

$$\mathbf{A}(\lambda) \cdot \mathbf{C} = 0 \tag{10}$$

where $\mathbf{C}=\left[C_1,C_2,C_3,C_4\right]^T$ and $\mathbf{A}(\lambda)=[a_{kj}]_{4\times 4}$. For existence of non-trivial solution to the buckling load problem it is necessary that the determinant of the matrix $\mathbf{A}$ be equal to zero

$$\det \mathbf{A}(\lambda)=0 \tag{11}$$

Equation (11) is then solved numerically using an approximate method.

The following matrices are obtained for various boundary conditions:

– clamped-clamped beams

$$\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2\beta & -\beta & \delta \\ 1 & e^{-2\beta} & e^{-\beta}\cos\delta & e^{-\beta}\sin\delta \\ 0 & -2\beta e^{-2\beta} & -e^{-\beta}\mu_1 & e^{-\beta}\mu_2 \end{bmatrix} \tag{12}$$

– pinned-pinned beams

$$\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 4\beta^2 & \beta^2-\delta^2 & -2\beta\delta \\ 1 & e^{-2\beta} & e^{-\beta}\cos\delta & e^{-\beta}\sin\delta \\ 0 & 4\beta^2 e^{-2\beta} & e^{-\beta}\mu_3 & e^{-\beta}\mu_4 \end{bmatrix} \tag{13}$$

– clamped-pinned beams

$$\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2\beta & -\beta & \delta \\ 1 & e^{-2\beta} & e^{-\beta}\cos\delta & e^{-\beta}\sin\delta \\ 0 & 4\beta^2 e^{-2\beta} & e^{-\beta}\mu_3 & e^{-\beta}\mu_4 \end{bmatrix} \tag{14}$$

where

$$\begin{aligned} &\mu_1=\beta\cos\delta+\delta\sin\delta \quad \mu_3=\beta^2\cos\delta-\delta^2\cos\delta+2\beta\delta\sin\delta \\ &\mu_2=\delta\cos\delta-\beta\sin\delta \quad \mu_4=\beta^2\sin\delta-\delta^2\sin\delta-2\beta\delta\cos\delta \end{aligned} \tag{15}$$

3. Numerical computations

In this section we present some numerical results. For uniform homogeneous beams comparisons of the exact solution [2], calculated by Huang and Luo [4] and the results obtained here are presented in Table 1. The calculated results correspond to the previous research.

Table 1

Dimensionless critical buckling loads for a uniform homogeneous beam with various boundary conditions

B C	C-C	C-P	P-P
Exact [2]	39.4784	20.1907	9.8696
[4]	39.478418	20.190729	9.869604
Present results	39.478417	20.190666	9.869604

Table 2

Dimensionless critical buckling loads for functionally graded beams with various boundary conditions

β \ B C	C-C	C-P	P-P
2	37.254959	7.10874	4.489585
1.5	37.849138	9.981006	6.190828
1	38.610809	13.324839	7.944048
0.5	39.237105	16.856634	9.331192
0.1	39.468432	19.558037	9.847218
–0.1	39.468432	20.801314	9.847218
–0.5	39.237105	22.991997	9.331192
–1	38.610809	25.135859	7.944048
–1.5	37.849138	26.750393	6.190828
–2	37.254959	28.119721	4.489585

Dimensionless critical buckling loads for three cases under consideration: clamped beams, simply-supported beams and clamped-pinned beams, were calculated for several various values of β. The obtained results are tabulated in Table 2. It can be observed that for clamped-pinned beams an increase of the value of β causes a decrease of the critical buckling load. Moreover, it is easily noticed that values of critical loads for clamped-clamped beams are symmetric, i.e. for opposite values of β we have the same results. This property also occurs for pinned-pinned beams. This is due to the fact that the boundary conditions at both ends are the same.

Conclusions

In this contribution we have shown an approach to determine the critical load of buckling of axially functionally graded beams subjected to a distributed axial load. Examples of computing buckling loads for three types of boundary conditions have been presented. Numerical examples show that the critical buckling loads of a homogeneous beam calculated by the proposed approach are in good agreement with those available in literature.

References

[1] Kukla S., Skalmierski B., The effect of axial loads on transverse vibrations of an Euler-Bernoulli beam, J. Theor. Appl. Mech. 1993, 31(2), 413-430.

[2] Wang C.M., Wang C.Y., Reddy J.N., Exact Solutions for Buckling of Structural Members, CRC Press, LLC, Florida 2005.

[3] Singh K.V., Li G., Buckling of functionally graded and elastically restrained non-uniform columns, Composites B 2009, 40, 393-403.

[4] Huang Y., Luo Q.Z., A simple method to determine the critical buckling loads for axially inhomogeneous beams with elastic restraint, Comput. Math. Appl. 2011, 61, 2510-2517.

[5] Shahba A, Rajasekaran S., Free vibration and stability of tapered Euler-Bernoulli beams made of axially functionally graded materials, Appl. Math. Model. 2012, 36, 3094-3111.

[6] Shahba A., Attarnejad R., Tavanaie Marvi M., Hajilar S., Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions, Composites B 2011, 42, 801-808.