Journal of Applied Mathematics and Computational Mechanics 2013, 12(4), 59-65

THERMALLY INDUCED VIBRATION OF A CANTILEVER BEAM WITH PERIODICALLY VARYING INTENSITY OF A HEAT SOURCE

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Abstract. In this paper an exact solution to the problem of the thermally induced vibration of a cantilever beam is presented. It is assumed that on a part of the beam the surface acts as a periodically time-varying heat source. The changing of the beam temperature produces thermal stresses, which cause displacements of the beam. The vibration of the beam is governed by the Bernoulli-Euler equation which includes the variable thermal moment. The heat equation and the vibration problem are solved by using the Green's function method. The symbolic software *Mathematica* has been used to obtain the solution of the problem in an analytical form.

Keywords: *heat conduction, beam vibration, Green's function*

Introduction

The solution to a problem of thermally induced vibrations of a beam includes determination of the temperature distribution, the thermal moment and the displacement of this beam. The exact analytical solutions of the heat conduction problem and vibration problem, obtained in the form of infinite series, are used for numerical computation of the temperature and displacements of the beam. The thermally vibration of the beams were considered by many authors [1-5].

The problem of the thermal induced flutter of a spacecraft boom was investigated by Yu in paper [1]. The effect of viscoelastic damping on the stability of the boom motion has been studied. A solution of the problem of thermally induced vibration of a simply supported beam has been presented by Kidawa-Kukla [2]. The analytical form of the solution was obtained by applying of the Green's function method. The application of the Green's function to heat conduction problems is widely presented by Beck et al. in book [3]. To improve the computation of the Green's function solutions the use of a time portioning method is proposed. The Green's functions, properties are also used to determine solutions of the beam vibration problems. The method has been applied to solving the problem of transverse vibrations of a beam induced by a mobile heat source by Kidawa-Kukla in paper [4]. The application of the method to vibration problems is presented by Duffy in book [5].

The aim of this paper is to determine a solution to the problem of a thermally induced vibration of a cantilever beam. Periodically varying stream of heat subjected to a portion of the beam causes changes in the temperature and produces thermal stresses and displacement of this beam. The exact solution of the problem can be used in numerical analysis of thermal vibration of the beams.

1. Heat conduction problem

The heat conduction in a uniform beam (Fig. 1) is governed by the equation

$$\nabla^2 T + \frac{1}{k} q(x,y,t) = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $T(x,y,t)$ - temperature of the beam at the point (x,y) at time t, k - thermal conductivity, κ - thermal diffusivity and $q(x,y,t)$ represents a heat generation term. Equation (1) is complemented by initial and boundary conditions

$$T(x,y,0) = F_0(x,y) \tag{2}$$

$$T(0,y,t) = T(L,y,t) = 0 \tag{3}$$

$$\frac{\partial T}{\partial y}(x,0,t) = -\mu_0 \left[T_0 - T(x,0,t)\right] \tag{4}$$

$$\frac{\partial T}{\partial y}(x,h,t) = \mu_1 \left[T_1 - T(x,h,t)\right] \tag{5}$$

where $\mu_0 = \frac{\overline{\alpha}_0}{k}$, $\mu_1 = \frac{\overline{\alpha}_1}{k}$, $\overline{\alpha}_0$ and $\overline{\alpha}_1$ are the heat transfer coefficients.

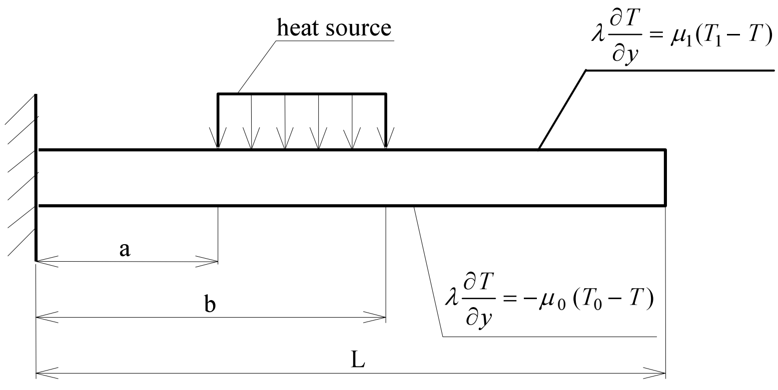

Fig. 1. A sketch of the beam considered

The energy generation function $q(x,y,t)$ is assumed in the form

$$q(x,y,t)=\theta(1+A\sin\nu t)Q(x)\delta(y) \tag{6}$$

where θ characterises the stream of heat, $\delta(\cdot)$ is the Dirac delta function, $Q(x)=1$ for $x\in[a,b]$ and $Q(x)=0$ for other values of x.

An analytical form of the solution to the problem (1-5) can be determined by using the Green's functions G_X^T and G_Y^T which correspond to the problems of heat conduction in the x and y direction, respectively. The obtained temperature distribution in the beam can be written in the following form [3]

$$\begin{aligned}T(x,y,t)&=\int_{\xi=0}^{L}\int_{\eta=0}^{h}G_X^T(x,t;\xi,0)\,G_Y^T(y,t;\eta,0)\,F_0(\xi,\eta)d\xi\,d\eta\\&+\frac{\kappa}{k}\int_{\tau=0}^{t}\int_{\xi=0}^{L}\int_{\eta=0}^{h}G_X^T(x,t;\xi,\tau)\,G_Y^T(y,t;\eta,\tau)\,q(\xi,\eta,\tau)d\xi\,d\eta\\&+\frac{\kappa}{k}\int_{\tau=0}^{t}\int_{\eta=0}^{h}G_X^T(x,t;\xi,\tau)\,G_Y^T(x,t;0,\tau)\,T_0(\xi,\tau)d\xi\,d\eta\\&+\frac{\kappa}{k}\int_{\tau=0}^{t}\int_{\eta=0}^{h}G_X^T(x,t;\xi,\tau)\,G_Y^T(x,t;h,\tau)\,T_1(\xi,\tau)d\xi\,d\eta\end{aligned} \tag{7}$$

The Green's functions G_X^T and G_Y^T are given in the book by Beck [3].

Taken into account the Green's functions in equation (7), we obtain

$$\begin{aligned}T(x,y,t)&=\frac{4}{hL}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{I_{mn}}{q_n}\exp\left(-\kappa\left(\alpha_m^2+\beta_n^2\right)t\right)\sin(\alpha_m x)\psi_n(y)+\\&+\frac{4\kappa\theta}{hL\,k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\alpha_m\,q_n}K_{mn}(t)\sin(\alpha_m x)\psi_n(y)\psi_n(0)(\cos(\alpha_m b)-(\alpha_m a))+\\&+\frac{4\kappa}{hL\,k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{q_n}J_{mn}(t)\sin(\alpha_m x)\psi_n(y)\end{aligned} \tag{8}$$

where

$$I_{mn}=\int_{\xi=0}^{x=L}\int_{\eta=0}^{y=h}\sin(\alpha_m\xi)\psi_n(\eta)\,F_0(\xi,\eta)d\xi\,d\eta \tag{8a}$$

$$J_{mn}(t)=\int_{\tau=0}^{t}\int_{\eta=0}^{h}\exp\left(-\gamma_{mn}(t-\tau)\right)\sin(\alpha_m\xi)\left(\psi_n(0)T_0(\xi,\tau)+\psi_n(h)T_1(\xi,\tau)\right)d\tau\,d\xi \tag{8b}$$

$$K_{mn}(t)=\frac{1-\exp(-\gamma_{mn}t)}{\gamma_{mn}}+\frac{A\left(v\exp(-\gamma_{mn}t)-v\cos(vt)+\gamma_{mn}\sin(vt)\right)}{v^2+\gamma_{mn}^2} \tag{8c}$$

and

$$\alpha_m=\frac{m\pi}{L},\qquad q_n=\left(\beta_n^2+\mu_0^2\right)\left[1+\frac{\mu_1^2}{\beta_n^2+\mu_1^2}\right]+\mu_0^2,\qquad \psi_n(y)=\beta_n\cos\beta_n y+\mu_0\sin\beta_n y,$$

$\gamma_{mn}=\kappa\left(\alpha_m^2+\beta_n^2\right)$, $\mu_0=\dfrac{h_1}{k}$, $\mu_1=\dfrac{h_2}{k}$. The values β_n are roots of equation

$$\left(\beta_n^2-\mu_0\mu_1\right)\sin(\beta_n h)-\beta_n(\mu_0+\mu_1)\cos(\beta_n h)=0 \tag{9}$$

2. The problem of the thermally induced vibration of the beam

The thermally induced vibration of the considered beam without the internal damping is governed by the biharmonic differential equation

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=M(x,t) \tag{10}$$

where EI is the bending rigidity, A is the area of the cross-section, w is the lateral beam deflection, x is the distance along the length of the beam and t denotes time. The thermal moment $M(x,t)$ is defined by

$$M(x,t)=\alpha bE\int_0^h\left(y-\frac{h}{2}\right)\frac{\partial^2 T(x,y,t)}{\partial x^2}\,dy \tag{11}$$

where α is the coefficient of the thermal expansion, E is Young's modulus. The equation (10) is complemented by zero-value initial conditions

$$w(x,0)=\frac{\partial w}{\partial t}(x,0)=0 \tag{12}$$

and the boundary conditions corresponding to the cantilever beam

$$w(0,t)=\frac{\partial w}{\partial x}(0,t)=0,\qquad \frac{\partial^2 v}{\partial x^2}(L,t)=\frac{\partial^3 v}{\partial x^3}(L,t)=0 \tag{13}$$

Substituting the temperature $T(x,y,t)$ given by equation (8) into equation (11), we obtain the thermal moment in the form

$$M(x,t) = -\frac{4}{hL}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{I_{mn}\alpha_m^2 p_n}{q_n}\exp(-\gamma_{mn}t)\sin(\alpha_m x) +$$
$$-\frac{4\kappa\theta}{hLk}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m p_n}{q_n}K_{mn}(t)\sin(\alpha_m x)\psi_n(0)(\cos(\alpha_m b)-(\alpha_m a)) + \quad (14)$$
$$-\frac{4\kappa}{hLk}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m^2 p_n}{q_n}J_{mn}(t)\sin(\alpha_m x)$$

where

$$p_n = \frac{2\alpha bE}{\beta_n^2}(\vartheta_n\cos\vartheta_n - \sin\vartheta_n)(\beta_n\sin\vartheta_n - \mu_0\sin\vartheta_n)$$

and $\vartheta_n = \dfrac{h\beta_n}{2}$.

The solution to the problem (10)-(13) in an analytical form is obtained by using the properties of the Green's function G^w, which is a solution of the differential equation (10)

$$EI\frac{\partial^4 G^w}{\partial x^4} + \rho A\frac{\partial^2 G^w}{\partial t^2} = \delta(x-\xi)\,\delta(t-\tau) \quad (15)$$

This function satisfies the initial and boundary conditions analogous to the conditions for displacement function given by equations (12)-(13). The Green's function can be written in the following form

$$G^w(x,\xi,t-\tau) = \frac{1}{\Omega^4}H(t-\tau)\sum_{n=1}^{\infty}\frac{1}{\beta_n Q_n}\Phi_n(x)\Phi_n(\xi)\sin\beta_n(t-\tau) \quad (16)$$

where $\Omega^4 = \dfrac{\rho A}{EI}$, $\beta_n^2 = \dfrac{\lambda_n^4}{\Omega^4}$, λ_n are roots of equation: $\cos\lambda_n L\cosh\lambda_n L + 1 = 0$ and

$$Q_n = \frac{1}{4\lambda_n}\big[8L\lambda_n\sin\lambda_n L\sinh\lambda_n L + (2\lambda_n L - 3\sin 2\lambda_n L)\sinh^2\lambda_n L +$$
$$+ (4\lambda_n L + 2\lambda_n L\cos 2\lambda_n L + 3\sin 2\lambda_n L)\mathrm{tg}^2\lambda_n L\big]$$

$$\Phi_n(x) = (\sin\lambda_n x - \sinh\lambda_n x)(\cos\lambda_n L + \cosh\lambda_n L) +$$
$$-(\cos\lambda_n x - \cosh\lambda_n x)(\sin\lambda_n L + \sinh\lambda_n L)$$

The displacement function $w(x,t)$ may be written in the form

$$w(x,t)=\int_0^t\int_0^L M(\xi,\tau)G^w(x,\xi,t-\tau)\,d\xi\,d\tau \tag{17}$$

Using equations (14) and (16) in equation (17), we have

$$\begin{aligned} w(x,t) = &-\frac{4}{hL\Omega^4}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_m^2\,\Gamma_{jm}\,I_{mn}\,\frac{p_n}{\beta_j Q_j q_n}\Phi_j(x)\chi_{jmn}(t)+\\ &-\frac{4\kappa\theta}{hLk\Omega^4}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m p_n}{\beta_j Q_j q_n}\Gamma_{jm}\,\overline{K}_{mn}(t)\Phi_j(x)\psi_n(0)\big(\cos(\alpha_m b)-(\alpha_m a)\big)+\\ &-\frac{4\kappa}{hLk\Omega^4}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m^2 p_n}{\beta_j Q_j q_n}\Gamma_{jm}\,\overline{J}_{mn}(t)\Phi_j(x) \end{aligned} \tag{18}$$

where

$$\Gamma_{jm}=\int_0^L\Phi_j(\xi)\sin(\alpha_m\xi)\,d\xi,\qquad \chi_{jmn}(t)=\int_0^t\exp(-\gamma_{mn}\tau)\sin\beta_j(t-\tau)\,d\tau$$

$$\overline{K}_{mn}(t)=\int_0^t K_{mn}(\tau)\sin\beta_j(t-\tau)\,d\tau,\qquad \overline{J}_{mn}(t)=\int_0^t J_{mn}(\tau)\sin\beta_j(t-\tau)\,d\tau$$

The transverse vibration of the cantilever beam induced by the periodically changed heat source can be numerically investigated by using the equation (18).

Conclusions

In this paper, the problem of the transverse vibration of a cantilever beam induced by a periodically varying stream of heat was solved. The formulation of the problem was based on the differential equations of the heat conduction and the transverse vibration of the beam, which were complemented by suitable initial and boundary conditions. The temperature distribution and the transverse vibration of the beam in an analytical form were obtained by using the properties of the Green's function. The obtained solution can be used to numerical investigation of the beam vibration.

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