

HEAT CONDUCTION PROBLEM IN A TWO-LAYERED HOLLOW CYLINDER BY USING THE GREEN'S FUNCTION METHOD

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Abstract. In this paper, a solution of the heat conduction problem in a two-layered hollow cylinder by using the Green's function method is presented. The considerations concern the heat conduction in the radial direction while taking into account time-dependent boundary conditions. The continuity condition of the temperature at the dividing surface of the layers is written in the form of a Volterra integral equation.

Keywords: *heat conduction problem, two-layered cylinder, Green's function method*

Introduction

The problems of the heat conduction in hollow and solid composite cylinders, in recent years, were the subject of many studies (cf. refs [1-3]). A solution of the heat conduction problem in a composite multi-layered circular cylinder can be determined by using analytical methods. In the papers [1-2] the solution of the problem was obtained by using the Laplace transform. The inverse transform has been numerically determined. However, the calculations of the inverse transform lead often to numerical instabilities and that way the search for new methods to avoid the possible instability in numerical computation is purposeful. An analytical method for solving the heat conduction problem in a composite circular cylinder, which is an extension of an approach used in Cartesian coordinates, is proposed by Lu et al. in reference [3]. In the paper the temperature solution for n -layered finite cylinder by application of the Laplace transform has been obtained and the closed form solution as the real part of a function is given. The other approach to solve the problems is the use of the properties of Green's functions. The examples of applications of the method are given by Beck et al. in the book [4]. The temperature solutions in multi-layer bodies by using the Green's function method are presented by Haji-Sheikh and Beck in paper [5]. The study concerns the three-dimensional heat conduction with examples of a two-layer body in Cartesian coordinates.

In the present paper, the solution of the problem of heat conduction in radial direction in a two-layered circular cylinder is proposed. The solution is obtained by the use of the Green's function method.

1. Formulation of the problem

We consider a hollow two-layered cylinder in which the heat transfer in a radial direction is held (Fig. 1). The radial heat conduction in the layers are governed by the differential equations:

$$\frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r} \frac{\partial T_i}{\partial r} = \frac{1}{k_i} \frac{\partial T_i}{\partial t}, \quad r \in [r_{i-1}, r_i], \quad i = 1, 2 \quad (1)$$

where T_i is the temperature in i -th layer, r is the radial coordinate, t is time and k_i is the thermal diffusivity of i -th layer.

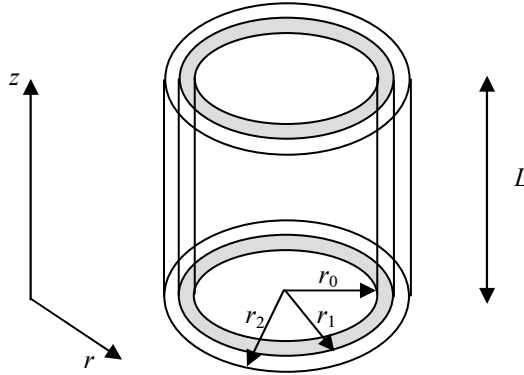


Fig. 1. The sketch of the hollow cylinder

We assume zero initial condition and boundary conditions which provide the free heat exchange at the inner ($r = r_0$) and outer ($r = r_2$) surfaces of the cylinder

$$T_i(0, r) = 0, \quad r \in [r_{i-1}, r_i], \quad i = 1, 2 \quad (2)$$

$$-\lambda_1 \frac{\partial T_1}{\partial r}(t, r_0) = -\alpha_+ (T_1(t, r_0) - T_+(t)) \quad (3)$$

$$-\lambda_2 \frac{\partial T_2}{\partial r}(t, r_2) = -\alpha_\infty (T_\infty(t) - T_2(t, r_2)) \quad (4)$$

where α_+ , α_∞ are the surface heat transfer coefficients, λ_1 , λ_2 are thermal conductivities and T_+ , T_∞ are the time-dependent boundary temperatures.

Moreover, at the contact surface of the layers, the continuity conditions are in force

$$T_1(t, r_1) = T_2(t, r_1) \quad (5)$$

$$\lambda_1 \frac{\partial T_1}{\partial r}(t, r_1) = \lambda_2 \frac{\partial T_2}{\partial r}(t, r_1) \quad (6)$$

2. Derivation of the temperature field

We determine the temperature T_1 and T_2 in the layers of the cylinder by using properties of Green's functions which correspond to the problem (1)-(6). The Green's functions G_1 and G_2 satisfy the auxiliary equation

$$\frac{\partial^2 G_i}{\partial r^2} + \frac{1}{r} \frac{\partial G_i}{\partial r} + \frac{1}{k_i r} \delta(r - r') \delta(t - \tau) = \frac{1}{k_i} \frac{\partial G_i}{\partial t}, \quad r \in [r_{i-1}, r_i], \quad i = 1, 2 \quad (7)$$

and the homogeneous boundary conditions and zero initial condition

$$-\lambda_1 \frac{\partial G_1}{\partial r}(t, r_0; \tau, r') = -\alpha_+ G_1(t, r_0; \tau, r') \quad (8)$$

$$\frac{\partial G_1}{\partial r}(t, r_1; \tau, r') = \frac{\partial G_2}{\partial r}(t, r_1; \tau, r') = 0 \quad (9)$$

$$-\lambda_2 \frac{\partial G_2}{\partial r}(t, r_2; \tau, r') = \alpha_\infty G_2(t, r_2; \tau, r') \quad (10)$$

$$G_i(0, r; \tau, r') = 0, \quad r \in [r_{i-1}, r_i], \quad i = 1, 2 \quad (11)$$

In equation (7) the $\delta(\cdot)$ denotes a Dirac delta function.

The derivation of the solution of the problem (7)-(11) we begin with use of the reciprocity relation [4]

$$G_i(t, r; \tau, r') = G_i(-\tau, r'; -t, r), \quad i = 1, 2 \quad (12)$$

Taking into account the relation (12) in equation (7), one obtains

$$\frac{\partial^2 G_i}{\partial r'^2} + \frac{1}{r'} \frac{\partial G_i}{\partial r'} + \frac{1}{k_i r'} \delta(r - r') \delta(t - \tau) = -\frac{1}{k_i} \frac{\partial G_i}{\partial \tau}, \quad i = 1, 2 \quad (13)$$

The temperature equations (1) we write also in terms of r' and τ as

$$\frac{\partial^2 T_i}{\partial r'^2} + \frac{1}{r'} \frac{\partial T_i}{\partial r'} = \frac{1}{k_i} \frac{\partial T_i}{\partial \tau}, \quad i = 1, 2 \quad (14)$$

Multiplying equation (13) by T_i and equation (14) by G_i and subtracting the obtained equations, one gives ($i = 1, 2$)

$$\frac{1}{r'} \frac{\partial}{\partial r'} \left(r' \frac{\partial T_i}{\partial r'} \right) G_i - \frac{1}{r'} \frac{\partial}{\partial r'} \left(r' \frac{\partial G_i}{\partial r'} \right) T_i - \frac{1}{k_i r'} \delta(r - r') \delta(t - \tau) T_i = \frac{1}{k_i} \frac{\partial (G_i T_i)}{\partial \tau} \quad (15)$$

Next, we integrate equation (15) with respect to r' from r_{i-1} to r_i ($i = 1, 2$) and integrate with respect to τ from 0 to t . The result is

$$\begin{aligned} \int_0^t \int_{r_{i-1}}^{r_i} \left[\frac{\partial}{\partial r'} \left(r' \frac{\partial T_i}{\partial r'} \right) G_i - \frac{\partial}{\partial r'} \left(r' \frac{\partial G_i}{\partial r'} \right) T_i - \frac{1}{k_i} \delta(r - r') \delta(t - \tau) T_i \right] dr' d\tau = \\ = \frac{1}{k_i} \int_0^t \int_{r_{i-1}}^{r_i} r' \frac{\partial (G_i T_i)}{\partial \tau} dr' d\tau, \quad i = 1, 2 \end{aligned} \quad (16)$$

Using now the properties of the Dirac delta function, we obtain

$$T_i(t, r) = k_i \int_0^t \int_{r_{i-1}}^{r_i} \left[\frac{\partial}{\partial r'} \left(r' \frac{\partial T_i}{\partial r'} \right) G_i - \frac{\partial}{\partial r'} \left(r' \frac{\partial G_i}{\partial r'} \right) T_i \right] dr' d\tau - \int_{r_{i-1}}^r r' (G_i T_i) \Big|_{\tau=0}^{\tau=t} dr \quad (17)$$

$i = 1, 2$

Integrating by parts the integral on the right-hand side of equation (17), and next using the boundary conditions (3)-(4) and (8)-(10), we have

$$T_i(t, r) = k_i \int_0^t r' \left[\frac{\partial T_i}{\partial r'} G_i - \frac{\partial G_i}{\partial r'} T_i \right]_{r'=r_{i-1}}^{r'=r_i} d\tau, \quad i = 1, 2 \quad (18)$$

Applying the boundary conditions (3) and (4) in equation (18), gives

$$T_1(t, r) = k_1 \int_0^t \left[r_1 \frac{\partial T_1(\tau, r_1)}{\partial r'} G_1(t, r; \tau, r_1) + r_0 \frac{\alpha_+}{\lambda_1} T_+(\tau) G_1(t, r; \tau, r_0) \right] d\tau \quad (19)$$

$$T_2(t, r) = k_2 \int_0^t \left[r_2 \frac{\alpha_\infty}{\lambda_2} T_\infty(\tau) G_2(t, r; \tau, r_2) - r_1 \frac{\partial T_2(\tau, r_1)}{\partial r'} G_2(t, r; \tau, r_1) \right] d\tau$$

The next step in the derivation of the temperature T_1 and T_2 is to determine the unknown derivatives $\frac{\partial T_1}{\partial r'}$ and $\frac{\partial T_2}{\partial r'}$ at $r' = r_1$. In order to do this, the continuity

conditions (5)-(6) will be used. Because from (6) we have $\frac{\partial T_2}{\partial r'} = \frac{\lambda_1}{\lambda_2} \frac{\partial T_1}{\partial r'}$, then using condition (5), we obtain

$$\begin{aligned} & k_1 \int_0^t \left[r_1 \frac{\partial T_1(\tau, r_1)}{\partial r'} G_1(t, r_1; \tau, r_1) + r_0 \frac{\alpha_+}{\lambda_1} T_+(\tau) G_1(t, r_1; \tau, r_0) \right] d\tau \\ & = k_2 \int_0^t \left[r_2 \frac{\alpha_\infty}{\lambda_2} T_\infty(\tau) G_2(t, r_1; \tau, r_2) - r_1 \frac{\lambda_1}{\lambda_2} \frac{\partial T_1(\tau, r_1)}{\partial r'} G_2(t, r_1; \tau, r_1) \right] d\tau \end{aligned} \quad (20)$$

The equation (20) can be written in the form:

$$\int_0^t K(t, \tau) y(\tau) d\tau = F(t) \quad (21)$$

where $y(\tau) = \frac{\partial T_1(\tau, r_1)}{\partial r'}$ and $K(t, \tau) = k_1 r_1 G_1(t, r_1; \tau, r_1) + k_2 r_1 \frac{\lambda_1}{\lambda_2} G_2(t, r_1; \tau, r_1)$ and

$$F(t) = \int_0^t \left[k_2 r_2 \frac{\alpha_\infty}{\lambda_2} T_\infty(\tau) G_2(t, r_1; \tau, r_2) - k_1 r_0 \frac{\alpha_+}{\lambda_1} T_+(\tau) G_1(t, r_1; \tau, r_0) \right] d\tau \quad (22)$$

Equation (21), as the Volterra integral equation of the first kind [6], will be solved by using a quadrature method. Finally, the temperatures T_1 and T_2 in the layers of the cylinder are given by equations:

$$T_1(t, r) = k_1 \int_0^t \left[r_1 y(\tau) G_1(t, r; \tau, r_1) + r_0 \frac{\alpha_+}{\lambda_1} T_+(\tau) G_1(t, r; \tau, r_0) \right] d\tau \quad (23)$$

$$T_2(t, r) = k_2 \int_0^t \left[r_2 \frac{\alpha_\infty}{\lambda_2} T_\infty(\tau) G_2(t, r; \tau, r_2) - r_1 \frac{\lambda_1}{\lambda_2} y(\tau) G_2(t, r; \tau, r_1) \right] d\tau$$

In order to compute the temperatures T_1 and T_2 we use a quadrature method to numerical calculation of integrals occurring in equations (21) and (23). After using the quadrature rule, the equation (21) gives

$$\sum_{j=1}^n A_j K(t, t_j \cdot h) y(t_j \cdot h) d\tau = F(t) \quad (24)$$

where t_j are the points of integration and A_j are the weights of the quadrature rule. Substituting successively: $t = t_i \cdot h$, $i = 1, \dots, n$, in equation (24) we obtain a system of equations with unknowns $y_i = y(t_i \cdot h)$, $i = 1, \dots, n$. The equation system can be written in the matrix form

$$\mathbf{A} \cdot \mathbf{Y} = \mathbf{F} \quad (25)$$

where $\mathbf{A} = [A_j K(t_i, t_j)]_{1 \leq i, j \leq n}$, $\mathbf{Y} = [y_1 \ y_2 \ \dots \ y_n]^T$, $\mathbf{F} = [F(t_1) \ F(t_2) \ \dots \ F(t_n)]^T$.

Next, the solution of the equation (25) is used in discretized form of equations (23) for the calculation of temperatures T_1, T_2 in the composite cylinder.

Conclusions

The solution of the heat conduction problem in a two-layered hollow cylinder by using the properties of the corresponding Green's functions is determined. The continuity condition on the interface of the cylinder layers leads to the Volterra integral equation of the first kind which can be numerically solved. Although the problem concerns the heat conduction in radial direction, the formulation and solution of the problem can be easily expanded to the heat conduction in axis direction of the cylinder.

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