QUATERNIONIC REGULAR FUNCTIONS IN THE SENSE OF FUETER AND FUNDAMENTAL 2-FORMS ON A 4-DIMENSIONAL ALMOST KÄHLER MANIFOLD

Wiesław Królikowski

Institute of Mathematics, Czestochowa University of Technology, Czestochowa, Poland
wieslaw_krolikowski@wp.pl

Abstract. A correspondence between quaternionic regular functions in the sense of Fueter and fundamental 2-forms on a 4-dimensional almost Kähler manifold is shown.

Keywords: *fundamental 2-form, almost Kähler manifold (complex analysis), Fueter regular function (quaternionic analysis).*

Introduction

It is interesting that using the properties of quaternionic regular functions in the sense of Fueter one can obtain significant results in complex analysis (see, e.g. [1, 2]). There are many amazing relations between quaternionic functions and some objects of complex analysis. This paper is devoted to showing one of them, namely that there is a correspondence between quaternionic regular functions in the sense of Fueter and fundamental 2-forms on a 4-dimensional almost Kähler manifold.

1. Basic notions

Let M^4 be a real C^∞-manifold of dimension 4 endowed with an almost complex structure J (i.e. J is a tensor field which is, at every point x of M^4, an endomorphism of the tangent space T_xM^4 so that $J^2 = -Id$, where Id denotes the identity transformation of T_xM^4) and a Riemannian metric g. If the metric g is invariant under the action of the almost complex structure J, i.e.

$$g(JX, JY) = g(X, Y)$$

for any vector fields X and Y on M^4, then (M^4, J, g) is called an *almost Hermitian manifold*.

Define the fundamental 2-form Ω by

$$\Omega(X, Y) := g(X, JY).$$

An almost Hermitian manifold (M^4, J, g, Ω) is said to be *almost Kähler* if Ω is a closed form, i.e.

$$d\Omega = 0.$$

Let us denote by the same letter the matrix Ω with respect to the coordinate basis. The matrix Ω is skew-symmetric so it can look as follows:

$$\Omega = \begin{pmatrix} 0 & \alpha & -\beta & \gamma \\ -\alpha & 0 & \eta & \delta \\ \beta & -\eta & 0 & \rho \\ -\gamma & -\delta & -\rho & 0 \end{pmatrix}.$$

Remark 1.1. We have

$$det\,\Omega = (\alpha\rho + \beta\delta + \gamma\eta)^2.$$

If Ω is a closed form ($d\Omega = 0$) then, using the following formula (see, e.g. [3], p. 36):

$$\begin{aligned} &d\Omega(X, Y, Z) \\ &\quad = \frac{1}{3}\{X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) \\ &\quad - \Omega([X, Y], Z) - \Omega([Z, X], Y) - \Omega([Y, Z], X)\}, \end{aligned}$$

where [,] denotes the Lie bracket, we obtain that the condition $d\Omega = 0$ is equivalent to the following system of first order partial differential equations:

$$\begin{aligned} \partial_w\eta + \partial_x\beta + \partial_y\alpha &= 0, \\ \partial_w\delta - \partial_x\gamma + \partial_z\alpha &= 0, \\ \partial_w\rho - \partial_y\gamma - \partial_z\beta &= 0, \\ \partial_x\rho - \partial_y\delta + \partial_z\eta &= 0, \end{aligned} \tag{1.1}$$

where (w, x, y, z) denote the coordinates in $\mathbf{R}^4$.

2. Preliminaries

Let **H** denote the set of quaternions. **H** is a 4-dimensional division algebra over **R** (real numbers) with basis $1, i, j, k$, where 1 is the identity and the quaternionic units i, j, k satisfy the conditions:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k,\ jk = i,\ ki = j.$$

(The quaternionic multiplication is not commutative but it is associative.)

A typical element (quaternion) q of **H** can be written as:

$$q = w + ix + jy + kz, \qquad w, x, y, z \in \mathbf{R}.$$

The conjugate of q is defined by

$$\overline{q} := w - ix - jy - kz$$

and the modulus (norm) by

$$||q||^2 := q \cdot \overline{q} = \overline{q} \cdot q = w^2 + x^2 + y^2 + z^2.$$

The norm can be used to express the inverse element: for $q \in \mathbf{H},\ q \neq 0$ we have

$$q^{-1} = \frac{\overline{q}}{||q||^2}.$$

The following relation is easy to check:

$$\overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1}, \qquad q_1, q_2 \in \mathbf{H}.$$

3. Fueter's regular functions

Denote by **H** the skew field of quaternions.

Let $U \subseteq$ **H** be an open set. A function $F : \mathbf{H} \supseteq U \rightarrow \mathbf{H}$ of the quaternionic variable $q = w + ix + jy + kz,\ (i, j, k$ - the quaternionic units) can be written as:

$$F = F_o + iF_1 + jF_2 + kF_3,$$

where F_o, F_1, F_2 and F_3 are real functions of 4 real variables w, x, y, z.

F_o is called the *real part* of F and $iF_1 + jF_2 + kF_3$ - the *imaginary part* of F.

In [4] Fueter introduced the following operator:

$$\overline{\partial}_{left} := \frac{1}{4}(\frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}).$$

Definition 3.1 ([4]). A quaternionic function $F : \mathbf{H} \supseteq U \rightarrow \mathbf{H}$ is said to be *left regular (in the sense of Fueter)* if it is differentiable in the real variable sense and satisfies the condition:

$$\overline{\partial}_{left} \cdot F = 0,$$

where the "$\cdot$" denotes the quaternionic multiplication.

The above condition can be rewritten in the following form:

$$
\begin{aligned}
(\partial_w + i\partial_x + j\partial_y + k\partial_z)\cdot(F_o + iF_1 + jF_2 + kF_3)& \\
= \partial_w F_o - \partial_x F_1 - \partial_y F_2 - \partial_z F_3& \\
+ i(\partial_w F_1 + \partial_x F_o + \partial_y F_3 - \partial_z F_2)& \\
+ j(\partial_w F_2 - \partial_x F_3 + \partial_y F_o + \partial_z F_1)& \\
+ k(\partial_w F_3 + \partial_x F_2 - \partial_y F_1 + \partial_z F_o) = 0.&
\end{aligned} \tag{3.1}
$$

Note that the last equation is equivalent to the following system of equations:

$$
\begin{aligned}
\partial_w F_o - \partial_x F_1 - \partial_y F_2 - \partial_z F_3 = 0,& \\
\partial_w F_1 + \partial_x F_o + \partial_y F_3 - \partial_z F_2 = 0,& \\
\partial_w F_2 - \partial_x F_3 + \partial_y F_o + \partial_z F_1 = 0,& \\
\partial_w F_3 + \partial_x F_2 - \partial_y F_1 + \partial_z F_o = 0.&
\end{aligned} \tag{3.2}
$$

There are many examples of left regular functions. Many papers have been devoted to studying the properties of those functions (see e.g. [2]). One has found the quaternionic generalizations of the Cauchy theorem, the Cauchy integral formula, Taylor series in terms of special polynomials etc.

4. Fundamental 2-forms associated with the Fueter's regular functions

THEOREM 4.1.

a) To any quaternionic function F of the form

$$F = Ai + Bj + Ck,$$

which is left regular in the sense of Fueter one can associate a skew-symmetric 4×4-matrix of the form:

$$
\Omega_F := \begin{pmatrix} 0 & C & -B & A \\ -C & 0 & A & B \\ B & -A & 0 & C \\ -A & -B & -C & 0 \end{pmatrix}. \tag{4.1}
$$

The 2-form Ω_F is closed: $d\Omega_F = 0$.

b) Conversely, to any skew-symmetric, 4×4-matrix Ω of the form (4.1) which is a closed 2-form one can associate univocally a quaternionic function:

$$F_\Omega := Ai + Bj + Ck,$$

which is left regular in the sense of Fueter.

c) We have

$$det\ \Omega_F = (A^2+B^2+C^2)^2 = ||F_\Omega||^2.$$

d) Take two skew-symmetric, 4×4-matrices of the form (4.1):

$$\Omega_1 = \begin{pmatrix} 0 & C_1 & -B_1 & A_1 \\ -C_1 & 0 & A_1 & B_1 \\ B_1 & -A_1 & 0 & C_1 \\ -A_1 & -B_1 & -C_1 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & C_2 & -B_2 & A_2 \\ -C_2 & 0 & A_2 & B_2 \\ B_2 & -A_2 & 0 & C_2 \\ -A_2 & -B_2 & -C_2 & 0 \end{pmatrix},$$

then the products $\Omega_1\cdot\Omega_2$ and $\Omega_2\cdot\Omega_1$ are of the form (4.1) if and only if the following condition:

$$A_1A_2+B_1B_2+C_1C_2=0$$

is satisfied.

e) Take two quaternionic functions of the form:

$$F_1 := A_1i+B_1j+C_1k,$$
$$F_2 := A_2i+B_2j+C_2k,$$

then the products $F_1\cdot F_2$ and $F_2\cdot F_1$ are of the form:

$$Ai+Bj+Ck$$

if and only if the following condition:

$$A_1A_2+B_1B_2+C_1C_2=0$$

is satisfied.

f) If

$$\Omega = \begin{pmatrix} 0 & C & -B & A \\ -C & 0 & A & B \\ B & -A & 0 & C \\ -A & -B & -C & 0 \end{pmatrix} \neq 0$$

then

$$\Omega^{-1} = -\frac{1}{A^2+B^2+C^2}\cdot\begin{pmatrix} 0 & C & -B & A \\ -C & 0 & A & B \\ B & -A & 0 & C \\ -A & -B & -C & 0 \end{pmatrix}$$
$$= -\frac{1}{A^2+B^2+C^2}\cdot\Omega = -\frac{1}{\sqrt{det\ \Omega}}\Omega.$$

g) If

$$F = Ai + Bj + Ck \quad (F{\neq}0),$$

then

$$F^{-1} = \frac{\overline{F}}{||F||} = \frac{-(Ai + Bj + Ck)}{\sqrt{A^2 + B^2 + C^2}} = -\frac{1}{||F||}F.$$

P r o o f. This follows immediately from (1.1) and (3.2).

Take any matrix Ω_o of the form (4.1):

$$\Omega_o := \begin{pmatrix} 0 & C_o & -B_o & A_o \\ -C_o & 0 & A_o & B_o \\ B_o & -A_o & 0 & C_o \\ -A_o & -B_o & -C_o & 0 \end{pmatrix}.$$

Denote by $\mathbf{V}(\Omega_o)$ the set of all matrices Ω of the form (4.1) which satisfy the condition:

$$AA_o + BB_o + CC_o = 0,$$

then the algebraic structure $(\mathbf{V}(\Omega_o), +, \cdot)$ is a vector space over **R**.

Analogously, take any quaternionic function F_o of the form:

$$F_o := A_oi + B_oj + C_ok.$$

Denote by $\mathbf{V}(F_o)$ the set of all functions F of the form:

$$F := Ai + Bj + Ck,$$

which satisfy the condition:

$$AA_o + BB_o + CC_o = 0,$$

then the algebraic structure $(\mathbf{V}(F_o), +, \cdot)$ is a vector space over **R**.

PROPOSITION 4.1. The mapping

$$\mathbf{F} : \mathbf{V}(F_o){\to}\mathbf{V}(\Omega_o),$$

defined by

$$\mathbf{F}(Ai + Bj + Ck) := \begin{pmatrix} 0 & C & -B & A \\ -C & 0 & A & B \\ B & -A & 0 & C \\ -A & -B & -C & 0 \end{pmatrix},$$

is an isomorphism between the vector spaces $\mathbf{V}(F_o)$ and $\mathbf{V}(\Omega_o)$.

References

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