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NUMERICAL SCHEME FOR A TWO-TERM SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. A numerical scheme is constructed to solve two-term sequential fractional differential equations with the orders of Caputo derivatives in the range (0,1). The proposed method is based on a corresponding existence-uniqueness theorem and transformation of the SFDE into an equivalent fractional integral equation. Numerical solutions are compared to analytical ones in two cases. An example with multiple solutions is also discussed.

Introduction

The paper is devoted to the numerical method of solving a certain sequential fractional differential equation (SFDE). Such equations in a two-term version were studied in paper [1], where existence-uniqueness results were obtained both for solutions generated by the stationary function as well as for solutions to the initial value problem (IVP).

Let us observe that fractional differential equations (FDEs) are widely applied in modeling many problems in physics, engineering, bioengineering, control theory, mechanics and economics [2-7]. In 2010 FDE theory also became an established area of mathematics confirmed by the Mathematical Subject Classification scheme (MSC 2010). Many methods of solutions, both analytical and numerical, have been studied in literature (compare [7-15] and references given therein). We discuss in the paper a two-term nonlinear sequential fractional differential equation (SFDE). In such equations, the differential operators are compositions of Caputo derivatives. The exact solution obtained in [1] is given as a limit of iterations of a certain mapping possible to calculate explicitly only in a linear case. Thus, we propose to develop a numerical scheme to visualize and compare solutions.

The paper is organized as follows. In the next section we recall the basics of fractional calculus necessary to formulate and solve the problem. We also quote two existence-uniqueness results which are the basis of the numerical approach. This scheme is given in Section 2 together with examples of numerical solutions. Section 3 is devoted to the study of a special case of two-term SFDE, where multiple solutions occur. There the standard numerical approach fails to recover the nontrivial solution and a certain modification is proposed to overcome this difficulty. The improvement, however, requires the knowledge of analytical solutions. This example indicates that numerical schemes based solely on initial conditions can give an incorrect result and they should be extended to multiple solutions problems. The paper is closed with a short conclusion.

1. Preliminaries

In this section we recall the basic definitions and theorems from fractional calculus, which we shall apply to formulate and solve a two-term SFDE. The left-sided Riemann-Liouville integral and Caputo derivative are defined as follows [7, 16].

Definition 1.1. The left-sided Riemann-Liouville integral of order α, denoted as I_{0+}^{α}, is given by the following formula for $Re(\alpha) > 0$:

$$I_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u)du}{(t-u)^{1-\alpha}} \tag{1}$$

Definition 1.2. Let $\mathrm{Re}(\alpha) \in (n-1, n)$. The left-sided Caputo derivative of order α, denoted as ${}^{c}D_{0+}^{\alpha}$, is given by the formula:

$${}^{c}D_{0+}^{\alpha} := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(u)du}{(t-u)^{\alpha-n+1}} \tag{2}$$

We shall consider a two-term fractional differential equation in an arbitrary finite interval [0,b] including left-sided Caputo derivatives in a sequential form:

$$(D^{\alpha_2} - a_1 D^{\alpha_1}) f(t) = \psi(t, f(t)) \tag{3}$$

with

$$D^{\alpha_1} f(t) := {}^{c}D_{0+}^{\alpha_1} \tag{4}$$

$$D^{\alpha_2} f(t) := {}^{c}D_{0+}^{\alpha_1}\, {}^{c}D_{0+}^{\alpha_2-\alpha_1} f(t), \quad \alpha_2 > \alpha_1 \tag{5}$$

The full proof of the existence-uniqueness result for the general solution to equation (3) and for the initial value problem in case $\alpha_1, \alpha_2 \in (0,1)$ are discussed in paper [1].

In the transformation of the above equation, we shall apply the following composition rule for the Caputo derivative and Riemann-Liouville integral. This rule holds for any function continuous in interval [0,b] and we quote it after monographs [7, 16].

Property 1.3. Let $f \in C([0,b],R)$ and $\beta > \alpha.$ The following equalities hold for any point $t \in [0,b]$

$$ {}^{c}D^{\alpha}_{0+} I^{\alpha}_{0+} f(t) = f(t) \tag{6} $$

$$ {}^{c}D^{\alpha}_{0+} I^{\beta}_{0+} f(t) = I^{\beta-\alpha}_{0+} f(t) \tag{7} $$

Assuming nonlinear term ψ to be a jointly continuous function and using the above property, we reformulate equation (3) as follows

$$ {}^{c}D^{\alpha_1}_{0+}\, {}^{c}D^{\alpha_2-\alpha_1}_{0+} \left(f(t) - a_1 I^{\alpha_2-\alpha_1}_{0+} f(t) - I^{\alpha_2}_{0+} \psi(t, f(t)) \right) = 0 \tag{9} $$

provided $f \in C[0,b].$ The function in the brackets belongs to the kernel of sequential derivative D^{α_2}. Let us denote this function as φ_0 and write the corresponding equation for the stationary function

$$ D^{\alpha_2}(\phi_0(t)) = {}^{c}D^{\alpha_1}_{0+}\, {}^{c}D^{\alpha_2-\alpha_1}_{0+} (\phi_0(t)) = 0 \tag{10} $$

which leads to the explicit formula

$$ \varphi_0(t) = \sum_{j=0}^{n_{1,2}-1} \frac{c_j \cdot t^j}{\Gamma(j+1)} + \sum_{i=0}^{n_1-1} \frac{d_i \cdot t^{\alpha_2-\alpha_1+i}}{\Gamma(\alpha_2-\alpha_1+i+1)} \tag{11} $$

when the respective orders fulfill the conditions:

$$ \alpha_1 \in (n_1 - 1, n_1) \text{ and } \alpha_2 - \alpha_1 \in (n_{1,2} - 1, n_{1,2}) $$

Equation (9), rewritten using stationary function (15), becomes the fractional integral equation:

$$ f(t) = a_1 I^{\alpha_2-\alpha_1}_{0+} f(t) + I^{\alpha_2}_{0+} \psi(t, f(t)) + \varphi_0(t) \tag{12} $$

which coincides with the following fixed point condition

$$ f(t) = T_{\varphi_0} f(t) \tag{13} $$

for mapping T_{φ_0} generated by stationary function φ_0.

Assuming function $\psi \in C([0,b] \times R, R)$ and observing that stationary function φ_0 is continuous in interval [0,b], we conclude that mapping T_{φ_0} transforms any continuous function into its continuous image.

The presented transformation of the FDE given in (3) into fixed point condition (13) allows us to formulate the existence-uniqueness result for a solution to equation (3) (compare [1]).

Proposition 1.4. Let $\alpha_2 > \alpha_1$ and function $\psi \in C([0,b] \times R, R)$ fulfill the Lipschitz condition $\forall_{t \in [0,b],} \forall_{x,y \in R}$:

$$\left|\psi(t,x) - \psi(t,y)\right| \le M \cdot \left|x - y\right| \tag{14}$$

Then, each stationary function of derivative D^{α_2} , given in (11) and fulfilling condition $T_{\varphi_0}\varphi_0 \ne \varphi_0$, yields a unique solution of equation (3) in the space of functions continuous in interval [0,b]. Such a solution is a limit of the iterations of mapping T_{φ_0}:

$$f(t) = \lim_{k \to \infty} (T_{\varphi_0})^k \chi(t) \tag{15}$$

where χ is an arbitrary continuous function.

The solution given in Proposition 1.4 depends on $n_1 + n_{1,2}$ constants. It corresponds to the general solution known in the classical theory of differential equations. These constants can be fixed using initial or boundary conditions. In the next proposition we describe a solution to an IVP (initial value problem) in the case when orders $\alpha_1, \alpha_2 \in (0,1)$.

Proposition 1.5. Let the assumptions of Proposition 1.4 be fulfilled and $\alpha_1, \alpha_2 \in (0,1)$. Then, the unique solution of equation (3) obeying the initial conditions:

$$f(0) = w_0,\ {}^{c}D_{0+}^{\alpha_2 - \alpha_1} f(0) = w_1 \tag{16}$$

exists in the $C[0,b]$ space. Such a solution is a limit of the iterations of mapping T_{φ_0} generated by the following stationary function

$$\varphi_0(t) = w_0 + \frac{(w_1 - a_1 w_0) t^{\alpha_2 - \alpha_1}}{\Gamma(\alpha_2 - \alpha_1 + 1)} \tag{17}$$

2. Numerical scheme for two-term sequential FDE

For many science problems, where fractional differential equations (FDEs) are applied in modeling, coming up with the exact solution analytically is difficult. Therefore it is necessary to construct and use numerical methods.

In the paper we use a modification of the fractional Adams-Bashforth-Moulton method proposed by Diethlem in [8]. The Predictor-Corrector method is an algorithm that proceeds in two steps. First, the prediction step calculates a rough approximation of the desired quantity. Second, the corrector step refines the initial approximation using another means.

2.1. Adams-Bashforth-Moulton scheme for two-term SFDE

We consider the equation

$$(D^{\alpha_2} - a_1 D^{\alpha_1}) f(t) = \psi(t, f(t))$$

with initial conditions

$$f(0) = w_0,\ {}^{c}D_{0+}^{\alpha_2-\alpha_1} f(0) = w_1 \tag{18}$$

for $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_2 > \alpha_1$. Assume that we are working on a uniform grid

$$0 = t_0 < t_1 < \ldots < t_j < t_{j+1} < \ldots < t_N = b$$

with an integer N, $h = b/N$ and $t_j = jh$.

By applying Property 1.3 we can convert initial value problem (18) for differential equation (3) into equivalent integral equation (12):

$$f(t) = a_1 I_{0+}^{\alpha_2-\alpha_1} f(t) + I_{0+}^{\alpha_2} \psi(t, f(t)) + \varphi_0(t)$$

Using the rectangle rule, we can calculate approximation $f_{k+1}^{P} \approx f(t_{k+1})$

$$f_{k+1}^{P} = \frac{a_1}{\Gamma(\alpha_2-\alpha_1)} \sum_{j=0}^{k} b_{j,k+1}^{(\alpha_2-\alpha_1)} f_j + \frac{1}{\Gamma(\alpha_2)} \sum_{j=0}^{k} b_{j,k+1}^{(\alpha_2)} \psi(t_j, f_j) + \varphi_0(t_{k+1}) \tag{19}$$

where

$$b_{j,k+1}^{(\beta)} = \frac{h}{\alpha_2}\left((k+1-j)^{\beta} - (k-j)^{\beta}\right) \tag{20}$$

The resulting value, f_{k+1}^{P}, is called the predictor.

Now, for equation (12) we can write the indirect one step Adams-Moulton scheme, using the product trapezoidal quadrature formula to replace integrals. If we replace the value of f_{k+1} by f_{k+1}^{P} in node t_{k+1}, then we obtain the formula:

$$\begin{aligned} f_{k+1} = {} & \frac{a_1}{\Gamma(\alpha_2-\alpha_1)} \sum_{j=0}^{k} a_{j,k+1}^{(\alpha_2-\alpha_1)} f_j + a_{k+1,k+1}^{(\alpha_2-\alpha_1)} f_{k+1}^{P} + \\ & + \frac{1}{\Gamma(\alpha_2)} \sum_{j=0}^{k} a_{j,k+1}^{(\alpha_2)} \psi(t_j, f_j) + a_{k+1,k+1}^{(\alpha_2)} \psi(t_{k+1}, f_{k+1}^{P}) + \varphi_0(t_{k+1}) \end{aligned} \tag{21}$$

where

$$a_{j,k+1}^{(\beta)} = \begin{cases} \dfrac{h^{\beta}}{\beta(\beta+1)}(k^{\beta+1} - (k-\beta)(k+1)^{\beta}) & j = 0, \\ \dfrac{h^{\beta}}{\beta(\beta+1)}((k-j+2)^{\beta+1} + (k-j)^{\beta+1} & \\ \qquad - 2(k-j+1)^{\beta+1}) & 1 \le j \le k, \\ \dfrac{h^{\beta}}{\beta(\beta+1)} & j = k+1. \end{cases} \tag{22}$$

2.2. Numerical examples

As an example, we shall discuss three variants of equation (3). For equations discussed in the first and second example, we can determine the analytical solutions. In the third case we present only a numerical solution.

Example 2.1. Consider the following simple version of equation (3):

$$D^{\alpha_2} f(t) = t^{\beta} \tag{23}$$

with initial conditions

$$f(0) = 0, \ {}^{c}D_{0+}^{\alpha_2-\alpha_1} f(0) = 1 \tag{24}$$

for $\alpha_1 = 0.3$, $\alpha_2 = 0.7$, $\beta = 2$.

Using analytical manipulations, we obtain the exact solution:

$$f(t) = \frac{\Gamma(3)t^{2.7}}{\Gamma(3.7)} + \frac{t^{0.4}}{\Gamma(1.4)} \tag{25}$$

The function on the right-hand side of equation (23) clearly fulfills the Lipschitz condition. From formula (17), we determine the stationary function:

$$\varphi_0(t) = \frac{t^{0.4}}{\Gamma(1.4)} \tag{26}$$

We explicitly check condition $T_{\varphi_0}\varphi_0 \neq \varphi_0$:

$$T_{\varphi_0}\varphi_0(t) = I_{0+}^{\alpha_2}\left(\frac{t^{0.4}}{\Gamma(1.4)}\right) + \frac{t^{0.4}}{\Gamma(1.4)} \neq \frac{t^{0.4}}{\Gamma(1.4)} = \varphi_0(t)$$

Table 1 shows the run time and percent age of error dependent on the step size.

Table 1

Error of Adams-Bashforth-Moulton method for Example 3.1

Step size	Min Percent Error	Avg Percent Error	Max Percent Error	Run time
1	0.371402	2.66616	10.4469	0.015 s
1/10	0.00376718	0.0321951	0.120384	0.686 s
1/20	0.000943073	0.00813277	0.0304244	2.715 s
1/50	0.000151046	0.00131046	0.00490237	16.895 s
1/100	0.000037778	0.000328532	0.0012293	68.001 s

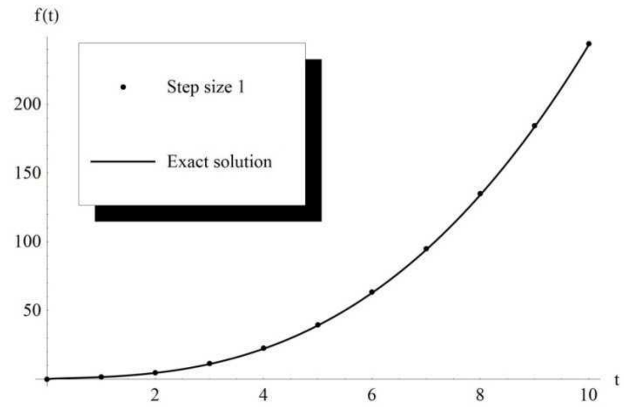

Fig. 1. Exact and numerical solution for Example 3.1

Analyzing Figure 1, we observe the high precision of numerical solutions even for the largest step size $h = 1$.

Example 2.2. Consider the linear case of equation (3):

$$D^{\alpha_2} f(t) = [f(t)]^{\beta} \tag{27}$$

with initial conditions

$$f(0) = 1,\ {}^{c}D_{0+}^{\alpha_2-\alpha_1} f(0) = 0 \tag{28}$$

for $\alpha_1 = 0.3,\ \alpha_2 = 0.5,\ \ \beta = 1.$

The exact solution of equation (27) is the following one-parameter Mittag-Leffler function [10]:

$$f(t) = E_{0.5,1}(t^{0.5}) \tag{29}$$

The function on the right-hand side of equation (27) fulfills the Lipschitz condition. From formula (17) we determine the stationary function:

$$\varphi_0(t) = 1 \tag{30}$$

We check whether the above stationary function fulfills condition $T_{\varphi_0}\varphi_0 \neq \varphi_0$:

$$T_{\varphi_0}\varphi_0(t) = I_{0+}^{\alpha_2} 1 + 1 \neq 1 = \varphi_0(t)$$

Table 2 shows the run time and percent age of error dependent on the step size.

Table 2

Error of Adams-Bashforth-Moulton method for Example 3.2

Step size	Min Percent Error	Avg Percent Error	Max Percent Error	Run time
1	60.456	81.5797	95.5736	0.016 s
1/10	3.6101	11.3189	19.4923	0.687 s
1/20	1.7513	4.76766	10.0607	2.637 s
1/50	0.45256	1.28408	5.93306	16.271s
1/100	0.162002	0.465502	4.12629	64.6 s
1/200	0.057387	0.166854	2.84665	267.99 s

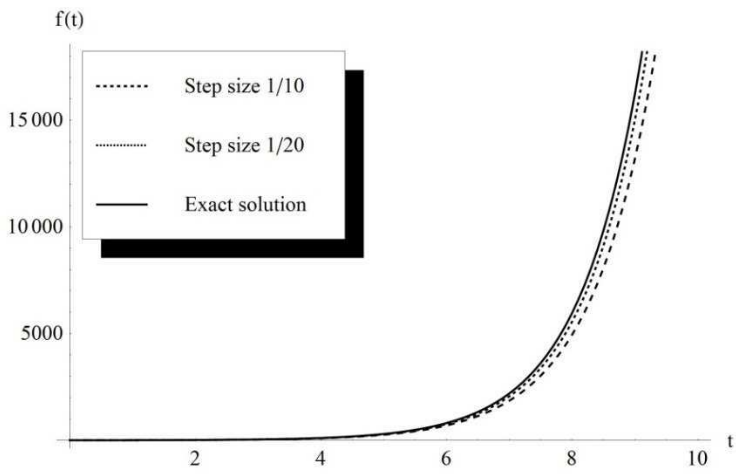

Fig. 2. Exact and numerical solution for Example 3.2

Example 2.3. Consider the following nonlinear case of equation (3):

$$(D^{\alpha_2} - a_1 D^{\alpha_1}) f(t) = \frac{f(t)\sin(t)}{(f(t))^2 + 1} \tag{31}$$

with initial conditions

$$f(0) = 1, \ {}^c D_{0+}^{\alpha_2 - \alpha_1} f(0) = 1 \tag{32}$$

Before we present the numerical results, we shall prove that the function on the right-hand side of equation (31) fulfills the Lipschitz condition:

$$|\psi(t,x)-\psi(t,y)| = \left|\frac{x\sin(t)}{x^2+1} - \frac{y\sin(t)}{y^2+1}\right| =$$

$$= |\sin(t)|\left|\frac{xy^2+x-yx^2-y}{(x^2+1)(y^2+1)}\right| \le \left|\frac{(y-x)(yx-1)}{(x^2+1)(y^2+1)}\right| =$$

$$= |x-y|\left|\frac{yx-1}{(y^2+1)(x^2+1)}\right| \le |x-y|\frac{|y||x|+1}{y^2x^2+y^2+x^2+1} \le$$

$$\le |x-y|\frac{y^2+x^2+1}{y^2x^2+y^2+x^2+1} \le |x-y|$$

We found Lipschitz constant M = 1 and this ends the proof.
The stationary function for equation (31) and initial conditions (32) looks as follows:

$$\varphi_0(t) = 1-(1-a_1)\frac{t^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2-\alpha_1+1)}$$

and it fulfills condition $T_{\varphi_0}\varphi_0 \ne \varphi_0$.

The complicated form of equation (31) makes it difficult to calculate an analytical solution. Therefore, we are looking only for approximate solutions. The results presented in Figures 3 and 4 were obtained for step size h = 1/50.

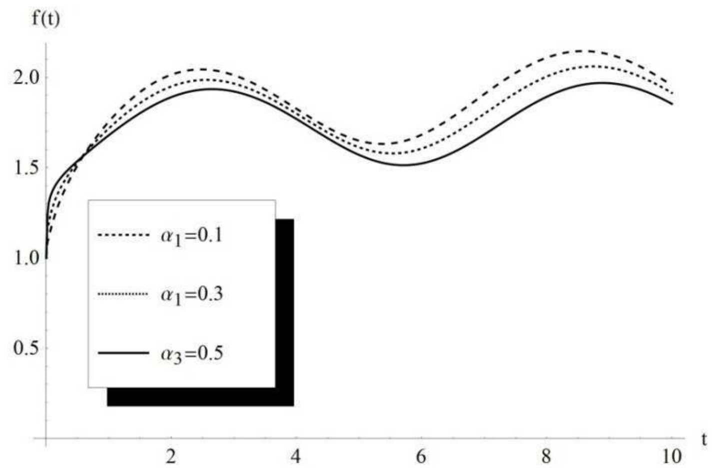

Fig. 3. Numerical solution for α2 = 0.8, a1 = −1

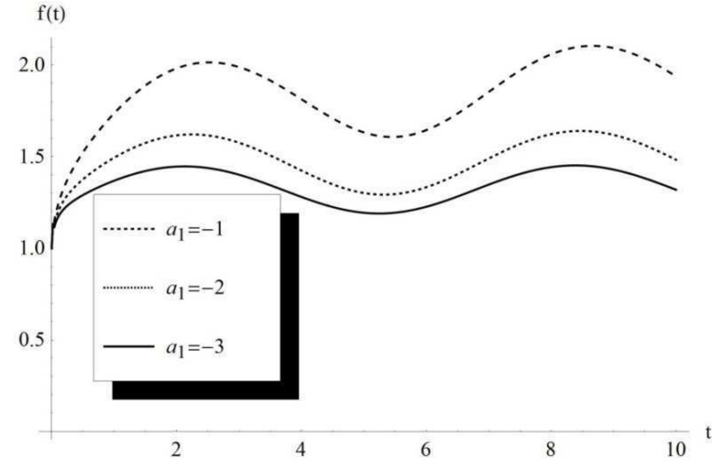

Fig. 4. Numerical solution for α1 = 0.2, α2 = 0.8

3. Example of IVP problem with $T_{\varphi_0}\varphi_0 = \varphi$

The initial value problem may have a unique solution, or more than one solution.
In this section, we consider a two-term SFDE for which the solution is determined by the stationary function with property $T_{\varphi_0}\varphi_0 = \varphi_0$. We derive the exact form of two solutions satisfying the given initial conditions. Next, we will show how to modify the Adams-Bashforth-Moulton algorithm to determine a non-zero solution.

3.1. Analytical solution

Consider equation:

$$D^{\alpha_2} f(t) = [f(t)]^{\beta} \tag{33}$$

with $\beta \neq 1$ and initial conditions

$$f(0) = 0,\ {}^{c}D_{0+}^{\alpha_2-\alpha_1} f(0) = 0 \tag{34}$$

For the above initial conditions, stationary function $\varphi_0(t) = 0$ and

$$T_{\varphi_0}\varphi_0(t) = I_{0+}^{\alpha_2}[\varphi_0(t)]^{\beta} + \varphi_0(t) = \varphi_0(t) \tag{35}$$

contradict condition $T_{\varphi_0}\varphi_0 \neq \varphi_0$, hence we conclude that equation (33) has more than one solution.
Let us assume that the solution of equation (33) is of the form

$$f(t) = Ct^{\gamma} \tag{36}$$

Incorporating function (36) into equation (34) we obtain the condition

$$D^{\alpha_2} Ct^{\gamma} = C^{\beta} t^{\gamma\beta} \tag{37}$$

and solving equation (37) we arrive at the relations

$$\gamma = \frac{\alpha_2}{1-\beta} \tag{38}$$

and

$$C = C^{\beta} \frac{\Gamma(\gamma+1-\alpha_2)}{\Gamma(\gamma+1)} \tag{39}$$

true for $C = 0$ and $C = [\Gamma(\gamma+1-\alpha_2) / \Gamma(\gamma+1)]^{1/(1-\beta)}$
Finally, we obtain two solutions to IVP (33), (34)

$$f(t) = \left[\frac{\Gamma\left(\frac{\alpha_2\beta}{1-\beta}+1\right)}{\Gamma\left(\frac{\alpha_2\beta}{1-\beta}+1+\alpha_2\right)} \right]^{\frac{1}{1-\beta}} t^{\frac{\alpha_2}{1-\beta}} \tag{40}$$

and $$f(t) = 0. \tag{41}$$

3.2. Numerical solution

For equation (33) with initial conditions (34) and $\alpha_2 = 0.7$, $\beta = 0.5$, the Adams-Bashforth-Moulton scheme described in (19), (21) yields only a zero solution. This is due to the construction of the predictor which depends solely on the initial conditions and function $\psi(t, \varphi_0(t)) \equiv 0$.

Table 3

Error of Adams-Bashforth-Moulton method

Step size	Avg Percent Error	Run time
1	7.26384	0.016 s
1/10	1.64206	0.812 s
1/20	0.96173	3.094 s
1/50	0.45928	19.156 s
1/100	0.25788	76.203 s

In the previous subsection, we derived the exact form of the non-vanishing solution. Now we use it to set an additional condition: $f_1 = f(h)$, where *f* is given by (40). Figure 5 and Table 3 show the nonzero solution of equation (33) determined by the Adams-Bashforth-Moulton scheme modified by the additional condition.

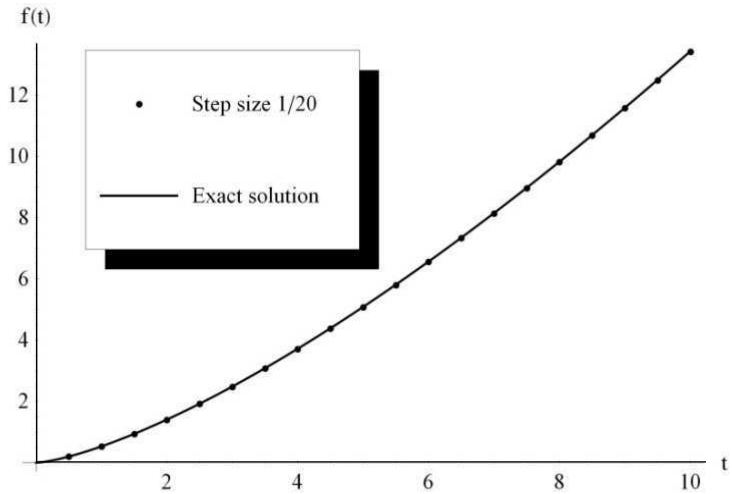

Fig. 5. Numerical solution for α2 = 0.7, β = 0.5

Conclusions

We constructed and discussed the numerical scheme to solve the initial value problem for a two-term SFDE with orders of derivatives fulfilling condition $\alpha_1, \alpha_2 \in (0,1)$. It is a variant of the Adams-Bashforth-Moulton method and as such, strongly depends on the properties of the stationary function generating the particular solution. The scheme was validated by comparison of the analytical and numerical solutions in two cases and then applied to a certain nonlinear SFDE. It will be further extended to multi-term SFDEs and improved with a more detailed error analysis and derivation of the experimental range of convergence (EOC). We should also point out that the example studied in Section 3 shows that in the case when a given SFDE has multiple solutions, the numerical scheme in the present form allows one only to derive a trivial solution which coincides with the stationary function. Thus, further investigation should also include extension of the method so as to derive non-trivial solutions.

Acknowledgments

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