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PROPERTIES OF PARTIAL ORDER IN GENERALISED INVERSE SEMIGROUPS

Jolanta Lipińska

Institute of Mathematics, Czestochowa University of Technology, Poland
jolanta.lipinska@im.pcz.pl

Abstract. In this paper we show that among the idempotent elements of a generalised inverse semigroup isomorphic to a pseudogroup of transformations on a topological space, is the largest element. We also show how to obtain the smallest element in some inverse semigroups.

Introduction

The notion of a pseudogroup was formed progressively together with the development of differential geometry. The first mathematicians who realized the classic notion of a group of transformations was not sufficient for the purposes of differential geometry were O. Veblen and J.H.C. Whitehead in 1932. Their definition was improved by J.A. Schouten and J. Haantjes in 1937, S. Gołąb in 1939 and C. Ehresmann in 1947. Gołąb's and Ehresmann's definitions are good enough to be used at present. It was shown in [1] that the axioms of Ehresmann's definition can be formulated in the equivalent way which simplify proofs. We used this definition in [2] to show that a group of transformation can be treated as a pseudogroup.

Main result

Let us recall the following version of Ehresmann's definition which is used in differential geometry and can be found in [3].

Definition 1. A pseudogroup of transformations on a topological space S is a set Γ of transformations satisfying the following axioms:

1° *Each* $f \in \Gamma$ *is a homeomorphism of an open set of* S *onto another open set of* S;

2° *If* $f \in \Gamma$, *then the restriction of* f *to an arbitrary open subset of the domain of* f *is in* Γ;

3° *Let* $U = \bigcup_i U_i$ *where each* U_i *is an open set of* S. *A homeomorphism* f *of* U *onto an open set of* S *belongs to* Γ *if the restriction of* f *to* U_i *is in* Γ *for every* i;

4° *For every open set* U *of* S, *the identity transformation of* U *is in* Γ;

5° *If* $f \in \Gamma$, *then* $f^{-1} \in \Gamma$;

6° *If* $f \in \Gamma$ *is a homeomorphism of* U *onto* V *and* $g \in \Gamma$ *is a homeomorphism of* Y *onto* Z *and if* $V \cap Y$ *is non-empty, then the homeomorphism* $g \circ f$ *of* $f^{-1}(V \cap Y)$ *onto* $g(V \cap Y)$ *is in* Γ.

We will also use the following definition introduced in [1].

Definition 2. *A non-empty set* Γ *of functions, for which domains* D_f *are arbitrary non-empty sets, will be called a pseudogroup of functions if it satisfies the following conditions*:

1° $f\left(D_f\right) \cap D_g \neq \varnothing \Rightarrow g \circ f \in \Gamma$ for $f, g \in \Gamma$,

2° $f^{-1} \in \Gamma$ for $f \in \Gamma$,

3° $\bigcup \Gamma' \in \Gamma$ *for* $\Gamma' \in \langle \Gamma \rangle$

where

$\langle \Gamma \rangle = \{\varnothing \neq \Gamma' \subset \Gamma : \bigcup \Gamma'$ *is a function and* $\bigcup (\Gamma')^{-1}$ *is a function*$\}$

and

$$(\Gamma')^{-1} = \left\{ f^{-1} : f \in \Gamma' \right\}$$

and f^{-1} *denotes an inverse relation.*

It was shown in [1] that if Γ is a pseudogrup, then $\left(\bigcup_{f \in \Gamma} D_f, \left\{ D_f : f \in \Gamma \right\} \cup \{\varnothing\} \right)$ is a topological space and Γ is an Ehresmann pseudogrup of transformations on this topological space. On the other hand, if Γ is an Ehresmann pseudogroup of transformations on a topological space S, then Γ is a pseudogrup of functions.

We will use the following definition which we can find in [4].

Definition 3. *We will say that* Γ *is a Schouten-Haantjes pseudogroup if it satisfies the following axioms:*

1° *If* $f \in \Gamma$, $g \in \Gamma$ *and* $g \circ f$ *is defined then* $g \circ f \in \Gamma$,

2° *If* $f \in \Gamma$ *and* f^{-1} *is defined then* $f^{-1} \in \Gamma$.

We will need the following definition which was introduced in [5].

Definition 4. *A generalized inverse semigroup is a partial groupoid* $(B, \bullet)$ *satisfying the following axioms:*

1° $a \bullet (b \bullet c) = (a \bullet b) \bullet c$

holds when one of the sides is defined;

2° *For every* $a \in B$ *there exists exactly one* $b \in B$ *such that*

$a \bullet (b \bullet a) = a$ and $b \bullet (a \bullet b) = b$

We will also need for the elements of a generalized inverse semigroup $(B, \bullet)$ the following definitions and denotations which were introduced in [5]. We will write ab insted of $a \bullet b$. For every $a \in B$ only one $b \in B$ from $2°$ of Definition 4 will be denoted by a' and called a generalised inverse element of a, $a'a$ will be called a right identity of a and aa' a left identity of a. It is obvious that a will then be a generalised inverse element of a', $a'a$ will be a left identity of a' and aa' a right identity of a'. If a is a right and left identity for all the elements of B, we say that a is an identity. We will say that $a \in B$ is an idempotent element when $aa = a.$ It was shown in [5] that $a = a'$ for an idempotent element a so it means that its generalised inverse element, right and left identity are all equal to a. It was also shown that the following relation

$$a \leq b \Leftrightarrow ba'a = a, \tag{1}$$

is a partial order in a generalized inverse semigroup. To prove it, we used a lemma saying that if a, b are idempotent elements, operation ab is commutative.

It was proved in [6] that we can obtain an inverse semigroup from every generalised inverse semigroup $(B, \bullet)$ joining an element $O \notin B$. Then $(B \cup \{O\}, *)$ is a semigroup where operation $*$ is defined in the following way

$$a * b = ab \text{ when operation } ab \text{ is defined} \tag{2}$$

$$O \text{ in the other case} \tag{3}$$

We will also use the theorem which was proved in [7] and says that if Γ is a pseudogroup of transformations on a topological space S then Γ is a generalised inverse semigroup with identity. Of course we can replace a pseudogroup of transformations by a pseudogroup of functions and the theorem will be true. It was shown in [7] that even a Schouten-Haantjes pseudogroup is a generalized inverse semigroup. As the definition of Schouten-Haantjes is more general, we can also say that a pseudogroup of functions is a generalized inverse semigroup. It was also proved in [8] that every generalised inverse semigroup is isomorphic to a Schouten-Haantjes pseudogroup.

Now we will formulate the problems. In which generalised inverse semigroups does the largest element among the idempotent elements exist? In which generalised inverse semigroups does the smallest element among the idempotent elements exist? We can formulate the following theorems.

Theorem 1. *When $(B, \bullet)$ is a generalised inverse semigroup isomorphic to a pseudogroup of transformations on a topological space, then there exists the largest element in the set of idempotent elements of $(B, \bullet)$.*

Proof. As we noticed above, $(B, \bullet)$ will be a generalised inverse semigroup with an identity. Let us denote this identity by $e.$ Let a be an idempotent element. As e is an identity, so the equality holds

$$ae = ea = a \tag{4}$$

Of course, e is its generalised inverse element, right and left identity simultaneously. As a is an idempotent element, it is also its generalised inverse element, right and left identity simultaneously. To show that $a \le e$, we need the equality

$$ea'a = a \tag{5}$$

but it holds because $a' = a$ and $aa = a$. That is required.

Theorem 2. When $(B, \bullet)$ is a generalised inverse semigroup, then there exists the smallest element in the semigroup $(B \cup \{O\}, *)$ obtained from $(B, \bullet)$ in the way shown earlier.

Proof. An inverse semigroup is also a generalised inverse semigroup so $(B \cup \{O\}, *)$ is also a generalised inverse semigroup. We can consider the relation of a partial order π defined as earlier

$$a \,\mathrm{p}\, b \Leftrightarrow b * (a' * a) = a \tag{6}$$

Of course, element O will be the smallest element because the equality holds

$$b * (O * O) = O \text{ and } O = O' \tag{7}$$

for all $b \in B \cup \{O\}$. That is required.

Conclusions

Of course we can replace a pseudogroup of transformations with a pseudogroup of functions and theorem 1 will be true. Element O is an idempotent element so it is the smallest among them but it is also the smallest among all the elements of $B \cup \{O\}$.

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