THE LOCAL AND GLOBAL INSTABILITY AND VIBRATION OF A NONLINEAR COLUMN SUBJECTED TO EULER'S LOAD

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Abstract. In this paper, the results of numerical studies on the local and global instability and vibration of a geometrically nonlinear column subjected to Euler's load are presented. The Hamilton principle was used to formulate the boundary problem. Due to the geometrical nonlinearity, the solution of the problem was performed by means of the perturbation method. The magnitude of the bifurcation load of the nonlinear column, the local and global instability regions and characteristic curves have also been presented.

Introduction

The local and global instability of geometrically nonlinear systems is a result of a comparative analysis on bifurcation load of geometrically nonlinear column and critical load value of corresponding linear column. The external load value at which the loss of rectilinear form of static equilibrium occurs is designated in the function of flexural rigidity factor. In numerical studies [1-3], it was proved that with lower flexural rigidity factor the local instability occurs for geometrically nonlinear column (the bifurcation load value of geometrically nonlinear column is smaller than critical load of corresponding linear system). In addition to theoretical considerations, the experimental studies were performed. In papers under the direction of Tomski [1, 2] the result of experiments on natural vibration frequency in a function of external load value were presented, as a confirmation of the local and global instability phenomenon. In the system presented in this paper the bifurcation load value may depend not only on flexural rigidity factor but also on localization of the pin and stiffness of connection of internal member. The formulation of the problem presented in this paper was obtained by means of Hamilton principle [4]. The solution of the problem due to nonlinearity of the column with rectilinear form of static equilibrium was performed by use of the small parameter method [5]. The solution of natural vibration of geometrically nonlinear columns by means of small parameter method was presented by Roordy and Chilvera [6], Tomski [1-3, 7, 8] and Przybylski [9]. The perturbation method used in order to solve the nonlinear differential equations was presented by Nayfeh [10].

1. Formulation of the problem

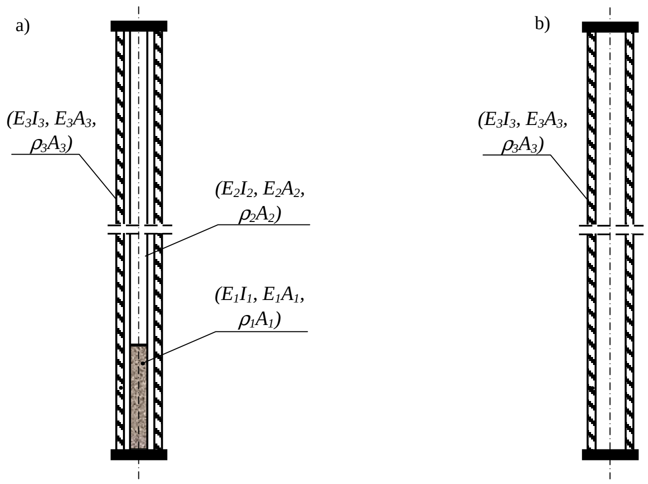

Fig. 1. Model of the column under consideration: a) geometrically nonlinear and b) geometrically linear

The physical model of considered geometrically nonlinear column (Fig. 1a) may be composed of two coaxial tubes (or tube and rod) or be a flat frame. In Figure 1b the corresponding linear column is presented. The geometrically linear system is devoided of the internal two segments member. Figure 2 shows the computational model of the considered nonlinear column, subjected to Euler's load.

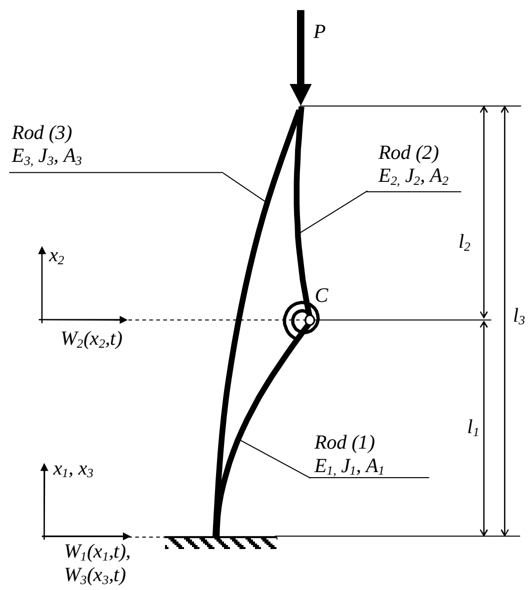

Fig. 2. The nonlinear column under consideration subjected to Euler's load

In the nonlinear column, rods (1) and (2) are joined together by a pin, strengthened by a rotational spring of stiffness *C*. The pin strengthened by a rotational spring represents in a physical model the connection of two rods. Rod (3) is discretely connected on both ends to the structure. The external load *P* is axially applied to the free end of the column (at the joint of rods (2) and (3)). The rods have a length l_1, l_2, l_3 respectively

The problem is formulated on the basis of Hamilton principle in the following form:

$$\delta\int_{t_1}^{t_2}(\mathrm{E}^k-\mathrm{E}^p)dt=0 \tag{1}$$

The kinetic E^k and potential E^p energy are expressed by the following formulas:

$$\mathrm{E}^k=\frac{1}{2}\sum_{i=1}^{3}\int_0^{l_i}\rho_i A_i\left(\frac{\partial W_i(x_i,t)}{\partial t}\right)^2 dx \tag{2}$$

$$\begin{aligned}\mathrm{E}^p=\frac{1}{2}\Bigg\{&\sum_{i=1}^{3}\int_0^{l_i}E_iJ_i\left[\frac{\partial^2 W_i(x_i,t)}{\partial x_i^{\,2}}\right]^2dx+\int_0^{l_i}E_iA_i\left[\frac{\partial U_i(x_i,t)}{\partial x_i}+\frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right]^2dx+\\&+C\left(\left.\frac{\partial W_2(x_2,t)}{\partial x_2}\right|_{x_2=0}-\left.\frac{\partial W_1(x_1,t)}{\partial x_1}\right|_{x_1=l_1}\right)^2\Bigg\}+PU_3(l_3,t)\end{aligned} \tag{3}$$

where: E_i - Young modulus, J_i - moment of inertia, A_i - cross section area, ρ_i - density of the material, *C* - rotational spring stiffness, *P* - external load.

The formulas (2) and (3) are subjected to the Hamilton principle (1). After performing variational and integration operation, and assuming that virtual displacement: longitudinal $\delta U_i(x_i,t)$ and transversal $\delta W_i(x_i,t)$ for $i=1,2,3$; are arbitrary and independent for $0 < x_i < l$ the following formulas were obtained:

– equation of motion in transversal direction

$$E_iJ_i\frac{\partial^4 W_i(x_i,t)}{\partial x_i^{\,4}}-E_iA_i\frac{\partial}{\partial x_i}\left[\left[\frac{\partial U_i(x_i,t)}{\partial x_i}+\frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right]\frac{\partial W_i(x_i,t)}{\partial x_i}\right]+\rho_iA_i\frac{\partial^2 W_i(x_i,t)}{\partial t^2}=0$$

$$i=1,2,3 \tag{4}$$

– equation of motion in longitudinal direction

$$U_i(x_i,t)=-\frac{S_i(t)x_i}{E_iA_i}-\frac{1}{2}\int_0^{x_i}\left[\frac{\partial W_i(x_i,t)}{\partial x_i}\right]^2dx_i \quad i=1,2,3 \tag{5}$$

– equation of immutability of strain along the length of the element

$$E_i A_i \frac{\partial}{\partial x_i}\left(\frac{\partial U_i(x_i,t)}{\partial x_i} + \frac{1}{2}\left[\frac{\partial W_i(x_i,t)}{\partial x_i}\right]^2\right) = 0\,, \quad i = 1,2,3 \tag{6}$$

After integration of equation (6), the axial force is defined as follows:

$$S_i(t) = -E_i A_i\left(\frac{\partial U_i(x_i,t)}{\partial x_i} + \frac{1}{2}\left[\frac{\partial W_i(x_i,t)}{\partial x_i}\right]^2\right), \quad i = 1,2,3 \tag{7}$$

The equation (4) after introducing (7) has the following form:

$$E_i J_i W_i^{IV}(x_i,t) + S_i(t) W_i^{II}(x_i,t) + \rho_i A_i \ddot{W}_i(x_i,t) = 0\,, \quad i = 1,2,3 \tag{8}$$

In equation (8) roman numerals define the derivatives with respect to space variable x_i and dots define the derivatives with respect to time t. Introducing into Hamilton principle the boundary conditions in the form:

$$W_1(0,t) = W_1^I(x_1,t)\big|_{x_1=0} = W_3(0,t) = W_3^I(x_3,t)\big|_{x_3=0} = U_1(0,t) = U_3(0,t) = 0$$

$$W_1(l_1,t) = W_2(0,t) \qquad W_2(l_2,t) = W_3(l_3,t)$$

$$W_2^I(x_2,t)\big|_{x_2=l_2} = W_3^I(x_3,t)\big|_{x_3=l_3} \qquad U_1(l_1,t) = U_2(0,t)$$

$$U_2(l_2,t) = U_3(l_3,t) \tag{9a-k}$$

the following natural boundary conditions were obtained:

$$E_2 J_2 W_2^{II}(x_2,t)\big|_{x_2=l_2} + E_3 J_3 W_3^{II}(x_3,t)\big|_{x_3=l_3} = 0$$

$$E_3 J_3 W_3^{III}(x_3,t)\big|_{x_3=l_3} + P W_3^I(x_3,t)\big|_{x_3=l_3} + E_2 J_2 W_2^{III}(x_2,t)\big|_{x_2=l_2} = 0$$

$$E_1 J_1 W_1^{III}(x_1,t)\big|_{x_1=l_1} + S_1 W_1^I(x_1,t)\big|_{x_1=l_1} - E_2 J_2 W_2^{III}(x_2,t)\big|_{x_2=0} - S_2 W_2^I(x_2,t)\big|_{x_2=0} = 0$$

$$-E_2 J_2 W_2^{II}(x_2,t)\big|_{x_2=0} + C\left[W_2^I(x_2,t)\big|_{x_2=0} - W_1^I(x_1,t)\big|_{x_1=l_1}\right] = 0$$

$$E_1 J_1 W_1^{II}(x_1,t)\big|_{x_1=l_1} - C\left[W_2^I(x_2,t)\big|_{x_2=0} - W_1^I(x_1,t)\big|_{x_1=l_1}\right] = 0$$

$$S_1 = S_2 \qquad S_1 + S_3 = P \tag{10a-g}$$

After introducing the following non-dimensional values:

$$d_i = \frac{l_i}{l},\; c_b = \frac{Cl}{\sum\limits_i E_i J_i},\; p = \frac{Pl^2}{\sum\limits_i E_i J_i},\; r_m = \frac{E_3 J_3}{E_2 J_2},\; r_w = \frac{E_2 J_2}{E_1 J_1},\; k_i(\tau) = \frac{S_i(t) l^2}{E_i J_i}$$

$$\xi_i = \frac{x_i}{l},\; \tau = \Omega t,\; w_i(\xi_i,\tau) = \frac{W_i(x_i,t)}{l},\; u_i(\xi_i,\tau) = \frac{U_i(x_i,t)}{l},\; \omega_i^2 = \Omega_i^2 \frac{\rho_i A_i l^4}{E_i J_i} \quad (11a\text{-}k)$$

the problem has been solved by means of the small parameter ε method. Whereby longitudinal and transversal displacements, axial force and vibration frequency of each rod are written in a power series

$$w_i(\xi,\tau) = \sum_{n=1}^{N} \varepsilon^{2n-1} w_{i2n-1}(\xi,\tau) + O(\varepsilon^{2N+1}) \qquad u_i(\xi,\tau) = u_{i0}(\xi) + \sum_{n=1}^{N} \varepsilon^{2n} u_{i2n}(\xi,\tau) + O(\varepsilon^{2N+1})$$

$$k_i(\tau) = k_{io} + \sum_{n=1}^{N} \varepsilon^{2n} k_{i2n}(\tau) + O(\varepsilon^{2N+1}) \qquad \omega_i^2 = \omega_{0i}{}^2 + \sum_{n=1}^{N} \varepsilon^{2n} \omega_{i2n}{}^2 + O(\varepsilon^{2N+1}) \quad (12a\text{-}d)$$

Magnitudes from equations (12a-d) are introduced into equation of motion, axial force and boundary conditions. Then, terms are grouped at the same power of the small parameter ε, which leads to infinite sequence of equations. The solution presented in this paper was obtained on the basis of system of equations with small parameter in the first power.

2. Results of numerical calculation and analysis

In the program of numerical calculations prepared on the basis of a mathematical model, the influence of selected physical and geometrical system parameters on first vibration frequency has been determined. In Figures 3a and 3b the relation between bifurcation load and vibration frequency of geometrically nonlinear and geometrically linear (*LC*) column has been presented, for different flexural rigidity factor r_m value. The results of numerical calculation are presented in the central localization of the pin which connects internal rods of the geometrically nonlinear column and in different rotational spring stiffness value.

The curve plotted in Figures 3a and 3b marked as *LC*, illustrates the change of vibration frequency as a function of external load for geometrically linear system. The other curves correspond to the geometrically nonlinear column. On the axis of ordinates - the axis of external load value, the bifurcation load points (black) for geometrically nonlinear system and critical load point (black/white) for geometrically linear system have been marked. Comparing the loads of both systems, it can be noted that there exists a rotational spring stiffness below which irrespective to flexural rigidity factor the bifurcation load of geometrically nonlinear column is smaller than critical load value of geometrically linear system - this is called the local instability. At higher spring stiffness the critical (bifurcation) load of the linear column is smaller than bifurcation load of the nonlinear system.

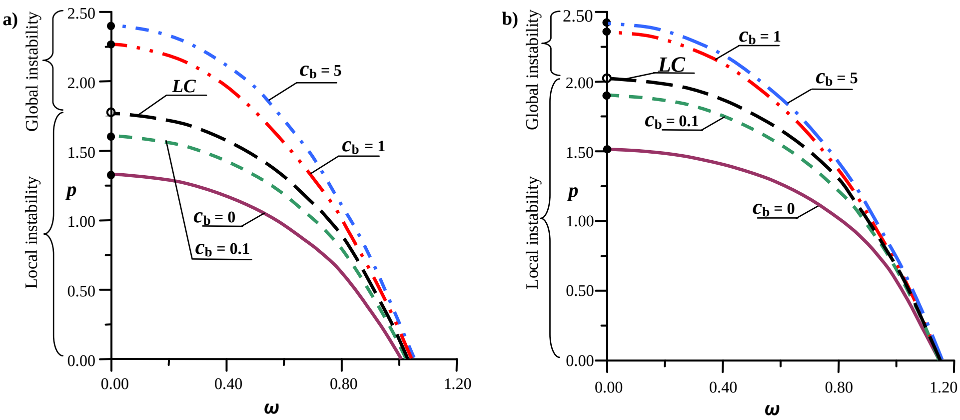

Fig. 3.The local and global instability on load - frequency plane in central localization of the pin and different flexural rigidity factor: a) $r_m = 0.4$, b) $r_m = 0.2$; $r_w = 1$

This phenomenon is called the global loss of the rectilinear form of static equilibrium. The increment of the rotational spring stiffness causes not only the increase of bifurcation load value but also vibration frequency of the nonlinear system. The bifurcation load and vibration frequency of the geometrically nonlinear system is highly dependent on rotational spring stiffness value. When the rotational spring stiffness is greater than 5 ($c_b > 5$), irrespective to localization of the pin, the change of vibration frequency and bifurcation load value is negligible. After change of the localization of the pin, it has been concluded that, then the pin is localized near the free end of the column, the influence of the stiffness connection of two member segment (rods 1 and 2) on bifurcation load value at which vibration frequency is equal to zero is reduced.

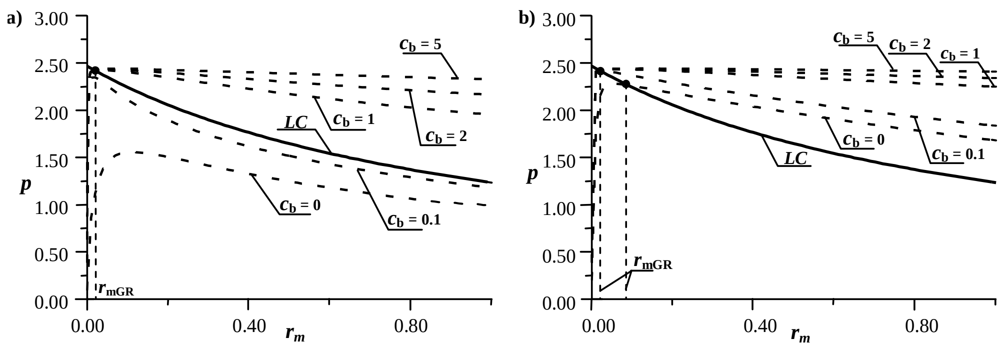

Fig. 4. The bifurcation load ***p*** as a function of flexural rigidity factor at different localization of the pin: a) $d_2 = 0.5$ and b) $d_2 = 0.7$; $r_w = 1$

In Figures 4a and 4b the relation between bifurcation load of geometrically nonlinear column and critical load of corresponding linear system as a function of flexural rigidity factor has been presented. The flexural rigidity factor is within $0 < r_m < 1$. The calculations have been performed for different localization of the pin which connects the internal rods and rotational spring stiffness constituting the geometrically nonlinear column.

The continuous curve plotted in Figures 4a and 4b marked as *LC* illustrates the change of maximum external load of the geometrically linear column and the dotted curves represents geometrically nonlinear system. Analyzing the maximum loads of both systems presented in Figures 4a and 4b it has been concluded that there is such a value of flexural rigidity factor r_m below which irrespectively from rotational spring stiffness and localization of the pin, the bifurcation (critical) load of the geometrically linear column is greater than bifurcation load of the geometrically nonlinear column. This phenomenon occurs for $r_m \in (0, r_{mGR})$. In this range of flexural rigidity factor the local loss of rectilinear form of static equilibrium occurs. The limitation value r_{mGR} designates the point of intersection of the geometrically linear and nonlinear load curve. In the range of flexural rigidity factor $r_m \in (r_{mGR}, 1)$ the global loss of rectilinear form of static equilibrium takes place. After moving the pin towards the free end of the column it has been noted that the influence of rotational spring stiffness on bifurcation load value is smaller. It also has been concluded that if the pin is localized near to the free end of the column the local instability region is decreased irrespectively from spring stiffness.

Conclusions

In this paper the divergence instability and vibration of the geometrically nonlinear cantilever column subjected to external load P with constant line of action has been presented. Bifurcation force value as dependent on the system parameters as well as local and global instability regions have been determined. After series of numerical calculations and analysis of the results, it has been concluded that the significant influence on type of instability has the localization of the pin which connects internal rods and stiffness of this connection. It was observed that, rotational spring stiffness above 5 causes insignificant change of vibration frequency and bifurcation load value. The change of localization of the pin towards to the free end of the column, causes stabilization of maximum load value apart from rotational spring stiffness. For the type of instability of the column, the stiffness and localization of connection of internal rods is responsible. It has been concluded that it is better to use the linear system when its critical load value is greater than the bifurcation load of the nonlinear system and inversely.

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