MEIJER G-FUNCTIONS SERIES AS EXACT SOLUTIONS OF A CLASS OF NON-HOMOGENEOUS FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. A fractional differential equation of real order α, containing variable coefficient t^{β} and a non-homogeneous term, is solved. The general solution is obtained as a sum of Meijer G-functions series determining the solution of a homogeneous counterpart of the considered equation and a series representing the particular solution of a non-homogeneous equation. The convergence of the respective series is analyzed in detail using theorems on properties of Meijer G-functions. As an example, two equations, with $\beta = 0$ and $\beta = \alpha/2$ are studied.

Introduction

Fractional calculus is an extension of classical calculus with integral and differential operators of non-integer order. It is used in mathematical modelling of various phenomena in physics, chemistry, mechanics, engineering, bioengineering and economics (compare monographs [1-3], papers [4-6] and the references therein).

A new class of integro-differential equations emerged as a result of the application of fractional calculus to the construction of models in many fields. The fractional differential equations theory became an important and interesting area of investigation. Many of the equations considered in literature are solved only numerically. Thus, procedures for exact analytical solutions are a subject still under investigation. The first monographs concerning these problems include exactness-uniqueness results as well as the application of integral transforms and operational procedures [7-9]. Here we shall consider a class of linear non-homogeneous equations of real non-integer order determined on the finite interval.

In our previous paper [10] a basic equation with a left-sided Riemann-Liouville derivative and variable coefficient was solved in its homogeneous version. The general solution of such an equation is the sum of component Meijer G-functions series. Their convergence and properties are thoroughly discussed in monograph [11]. Now we propose to consider a non-homogeneous equation of this type. We present the results obtained for equations with a non-homogeneous term in the form of an arbitrary linear combination of Meijer G-functions. These functions are determined by real vectors, but all the calculations can be easily extended to the case of complex vectors.

The paper is organized as follows. In the next section we recall basic definitions of fractional operators and their properties. Section 2 contains the main results enclosed in Proposition 2.2 and Corollary 2.3 on the exact general solutions of the considered non-homogeneous fractional differential equation. In Section 3 we give two examples of the application of the proposed method to the case of a constant coefficient and to the case when application of the reduction properties of Meijer G-functions simplifies the solution. The paper closes with concluding remarks and an appendix, where relevant properties of Fox and Meijer functions are recalled.

1. Fractional integrals and derivatives

In the paper we shall study a basic fractional differential equation with a left -sided Riemann-Liouville derivative. Let us recall the definitions of the left-sided integral and derivative of non-integer order $\alpha \in \mathbf{C}$ [7, 8, 12].

Definition 1.1

Let $Re(\alpha) > 0$. The left-sided Riemann-Liouville integral of order α is defined as follows:

$$\left(I_{0+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)\, dt}{(x-t)^{1-\alpha}}, \quad x > 0 \tag{1.1}$$

Using the above definition of a fractional integral, the Riemann-Liouville derivative is constructed.

Definition 1.2

Let $Re(\alpha) \in (n-1, n)$. The left-sided Riemann-Liouville derivative of order α is given by the following formula:

$$\left(D_{0+}^{\alpha} f\right)(x) =: \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\alpha} f\right)(x) \tag{1.2}$$

As we are investigating the fractional differential equations of real order, we shall assume what follows: that $\alpha \in (n-1, n)$. A detailed review of the properties of the introduced fractional operators can be found in monographs [7, 8, 12]. We quote here only one of the composition rules which we shall use in the solution procedure.

Lemma 1.3

(1) Let $\alpha \in (n-1, n)$. Then the following formula is valid for all $f \in C[0, b]$ and at any point $x \in [0, b]$

$$D^{\alpha}_{0+} I^{\alpha}_{0+} f(x) = f(x) \tag{1.3}$$

(2) Let $\alpha \in (n-1, n)$. Then the above composition rule is fulfilled for all $f \in C_{n-\alpha}[0, b]$ and at any point $x \in (0, b]$.

In the case of homogeneous equations considered in paper [10], it appeared that their solutions are Meijer G-functions series. Thence, we recall here the definition of this class of special functions.

Definition 1.4

Let $m, n, p, q \in \mathbf{N}_0$, $0 \le m \le q$, $0 \le n \le p$ and let $a_i, b_j \in \mathbf{C}$ be arbitrary complex numbers. The Meijer G-function $G^{m,n}_{p,q}(z)$ is given by the following formula [13]

$$G^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_i)_{1,p} \\ (b_j)_{1,q} \end{matrix} \right. \right] = G^{m,n}_{p,q}\left[z \left| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \right. \right] = \\ = \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + s) \prod_{i=1}^{n} \Gamma(1 - a_i - s)}{\prod_{i=n+1}^{p} \Gamma(a_i + s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - s)} z^{-s} ds \tag{1.4}$$

where $i = 1, 2, ..., p$; $j = 1, 2, ..., q$ and Γ is the Euler gamma function. Contour $\mathfrak{L}$ separates the poles of $\Gamma(b_j + s)$ and $\Gamma(1 - a_i - s)$ functions in the numerator of the above complex kernel in the integral.

Functions of this type were first introduced by Meijer in 1936 and then generalized by Fox in 1961. They are widely applied in probability theory, statistics, anomalous diffusion theory or fractional differential equations theory. We included in this paper some relevant properties of this class of functions - they are enclosed in Appendix A.

2. Non-homogeneous fractional differential equation with left-sided Riemann-Liouville derivative

Let us consider the non-homogeneous fractional differential equation of order $\alpha \in (n-1, n)$:

$$(x^{\beta} D^{\alpha}_{0+} - \lambda) f(x) = G^{m,n}_{p,q}(x) \quad x \in [0,b] \tag{2.1}$$

where $\beta \in \mathbf{R}$ and $G^{m,n}_{p,q}$ is the Meijer G-function defined by real vectors $\vec{a}$ and $\vec{b}$. The homogeneous version of such an equation was solved using the Mellin transform method in our previous paper [10]. The same method yields results in the case of an analogous equation with right-sided derivative [14] and in case of equation with symmetric or anti-symmetric fractional derivative [11,15]. The solution of a homogeneous counterpart of equation (2.1) is described in the following proposition.

Proposition 2.1

Let $\alpha \in (n-1, n)$, $\alpha > \frac{1}{2}$, $\{\alpha\} - \beta > 0$. Then equation

$$(x^{\beta} D^{\alpha}_{0+} - \lambda) f_0(x) = 0 \tag{2.2}$$

has in interval [0,b] a general solution in the form of

$$f_0(x) = \sum_{l=1}^{n} c_l f_0^l(x) \tag{2.3}$$

where component solutions f_0^l l=1,…,n are given by series

$$f_0^l(x) = b^{\alpha-l} \sum_{k=0}^{\infty} \left(\lambda\, b^{\alpha-\beta}\right)^k G^{1,\ k}_{k+1,k+1}\left[\frac{x}{b}\left|\begin{matrix}\vec{A}_{k,l}\\ \vec{B}_{k,l}\end{matrix}\right.\right] \tag{2.4}$$

with vectors $\vec{A}_{k,l}$, $\vec{B}_{k,l}$ defined as follows

$$\begin{aligned}
\vec{A}_{k,l} &= \left[\alpha \vec{e}_k + (\alpha - \beta)\vec{j}_k ; k(\alpha - \beta) + \alpha - l + 1\right] \\
\vec{B}_{k,l} &= \left[k(\alpha - \beta) + \alpha - l ; (\alpha - \beta)\vec{j}_k\right] \\
\vec{e}_k &= [1, ..., 1] \in \mathbf{R}^k \qquad \vec{j}_k = [0, 1, ..., k-1] \in \mathbf{R}^k
\end{aligned}$$

and $c_l \in \mathbf{R}$ arbitrary real coefficients.

Our aim now is to derive the general solution of the non-homogeneous version of equation (2.1). It will consist of the above general solution of a homogeneous problem and the particular solution of equation (2.1). Let us apply the composition rule from Lemma 1.3 in order to transform the studied equation into its integral form generating the particular solution:

$$(1 - \lambda I^{\alpha}_{0+} x^{-\beta}) f_s(x) = I^{\alpha}_{0+} x^{-\beta} G^{m,n}_{p,q}(x) \tag{2.5}$$

The solution of equation (2.5) can be written as a formal series

$$f_s(x) = \sum_{k=0}^{\infty} \lambda^k \left(I_{0+}^{\alpha} x^{-\beta}\right)^{k+1} G_{p,q}^{m,n}(x) \tag{2.6}$$

where

$$G_{p,q}^{m,n}(x) = G_{p,q}^{m,n}\left[x \middle| \begin{matrix} \vec{a} \\ \vec{b} \end{matrix} \right] \tag{2.7}$$

and vectors $\vec{a}, \vec{b}$ belong to the $\mathbf{R}^p$ and $\mathbf{R}^q$ spaces respectively.

To obtain an exact analytical form of particular solution (2.6), we should now calculate the fractional integrals on the right-hand side of formula (2.6) and analyze the convergence of the above series. Let us begin with $k = 0$ and calculate the parameters determining the properties of Meijer G-function $G_{p,q}^{m,n}$ [13]:

$$\Delta = q - p, \quad a^* = 2n + 2m - p - q \tag{2.8}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2} \tag{2.9}$$

$$\kappa = \begin{cases} -\min\limits_{1 \le i \le m} [b_i] & for \ \ m > 0 \\ -\infty & for \ \ m = 0 \end{cases} \tag{2.10}$$

$$\vartheta = \begin{cases} \min\limits_{1 \le j \le n} [1 - a_j] & for \ \ n > 0 \\ \infty & for \ \ n = 0 \end{cases} \tag{2.11}$$

We shall assume that the Meijer G-function in equation (2.1) fulfills the conditions

$$\kappa < \vartheta \quad \Delta\gamma + \mu < -1 \tag{2.12}$$

According to Theorem A.2, the first integral (when $k = 0$) exists, provided the conditions are valid:

$$\min_{1 \le j \le m} [b_j] - \beta > -1 \tag{2.13}$$

when $a^* > 0$ or $a^* = 0, \Delta \ge 0, \mu < -1$ and

$$\min_{1 \le j \le m} \left[b_j, \frac{\mu + \frac{1}{2}}{\Delta} \right] - \beta > -1 \tag{2.14}$$

when $a^* = 0,\ \Delta < 0, \mu < -1$. Both conditions (2.13) and (2.14) are a simple result of assumption $\kappa < 1 - \beta$. The result of fractional integration in this case is also the Meijer G-function

$$I_{0+}^{\alpha} x^{-\beta} G_{p,q}^{m,n}\left[x \left| \begin{matrix} \vec{a} \\ \vec{b} \end{matrix} \right.\right] = x^{\alpha-\beta} G_{p+1,q+1}^{m,n+1}\left[x \left| \begin{matrix} [\beta; \vec{a}] \\ [\vec{b}; \beta - \alpha] \end{matrix} \right.\right] \tag{2.15}$$

When k = 1 we obtain the following values of coefficients determining the result of the previous integration $I_{0+}^{\alpha} x^{-\beta} G_{p,q}^{m,n}$:

$$\Delta_1 = \Delta,\ \ a^{*1} = a^*,\ \ \mu_1 = \mu - \alpha,\ \ \kappa_1 = \kappa \tag{2.16}$$

$$\vartheta_1 = \begin{cases} \min\limits_{1 \le j \le n} [1 - a_j, 1 - \beta] & for\ \ n > 0 \\ 1 - \beta & for\ \ n = 0 \end{cases} \tag{2.17}$$

Let us note that for k = 1, the assumptions of Theorem A.2 are fulfilled, namely since we have $\{\alpha\} - \beta > 0$ and add $\kappa < \vartheta_1$, we obtain the following conditions valid:

$$\min_{1 \le j \le m} [b_j] - \beta + (\alpha - \beta) > -1 \tag{2.18}$$

in the case $a^* > 0$ or $a^* = 0,\ \Delta \ge 0,\ \mu < -1$ and

$$\min_{1 \le j \le m} \left[b_j, \frac{\mu - \alpha + \frac{1}{2}}{\Delta} \right] - \beta + (\alpha - \beta) > -1 \tag{2.19}$$

in the case $a^* = 0, \Delta < 0$, where $\mu < -1$.

From the above inequalities and from Theorem A.2 it follows that the second term in series (2.6) is of the form:

$$\left(I_{0+}^{\alpha} x^{-\beta}\right)^2 G_{p,q}^{m,n}(x) = \left(I_{0+}^{\alpha} t^{\alpha-2\beta} G_{p+1,q+1}^{m,n+1}(t)\right)(x) = x^{2(\alpha-\beta)} G_{p+2,q+2}^{m,n+2}\left[x \left| \begin{matrix} [2\beta - \alpha, \beta; \vec{a}] \\ [\vec{b}; \beta - \alpha, 2(\beta - \alpha)] \end{matrix} \right.\right] \tag{2.20}$$

Let us now consider arbitrary integer k. The parameters determining the properties of the respective Meijer G-function resulting from the (k–1)-th integration are as follows:

$$\Delta_k = \Delta,\ a^{*k} = a^*,\ \mu_k = \mu - k\alpha,\ \kappa_k = \kappa$$

$$\vartheta_k = \vartheta_1 = \begin{cases} \min\limits_{1\le j\le n} [1-a_j, 1-\beta] & for\ n>0 \\ 1-\beta & for\ n=0 \end{cases} \tag{2.21}$$

In this general case we shall calculate the fractional integral of order α of function $x^{-\beta}\left(I_{0+}^{\alpha}x^{-\beta}\right)^k G_{p,q}^{m,n}$. Similar to case k = 0,1 discussed previously, the assumptions of Theorem A.2 on fractional integration are fulfilled. The following inequalities are implied by condition $\kappa < \vartheta_1 = \vartheta_k$:

$$\min_{1\le j\le m}[b_j] - \beta + k(\alpha-\beta) > -1$$

provided $a^* > 0$ or $a^* = 0$, $\Delta \ge 0$, $\mu < -1$ and

$$\min_{1\le j\le m}\left[b_j, \frac{\mu - k\alpha + \frac{1}{2}}{\Delta}\right] - \beta + k(\alpha-\beta) > -1 \tag{2.25}$$

provided $a^* = 0$ and $\Delta < 0$, where $\mu < -1$. The result of the integration is the following Meijer G-function

$$\left(I_{0+}^{\alpha}x^{-\beta}\right)^k G_{p,q}^{m,n}(x) = \left(I_{0+}^{\alpha}t^{k\alpha-(k+1)\beta}G_{p+k,q+k}^{m,n+k}(t)\right)(x) = x^{(k+1)(\alpha-\beta)}G_{p+k+1,q+k+1}^{m,n+k+1}\left[x\left|\begin{matrix}\vec{E}_{k+1}\\ \vec{F}_{k+1}\end{matrix}\right.\right] \tag{2.26}$$

with vectors $\vec{E}_{k+1}, \vec{F}_{k+1}$ given by formulas

$$\vec{E}_{k+1} = \left[\beta\vec{e}_{k+1} + (\beta-\alpha)\vec{j}_{k+1}; \vec{a}\right]$$

$$\vec{F}_{k+1} = \left[\vec{b}; (\beta-\alpha)\vec{e}_{k+1} + (\beta-\alpha)\vec{j}_{k+1}\right]$$

We have explicitly calculated all the elements of the series describing the particular solution in terms of Meijer G-functions

$$f_s(x) = \sum_{k=0}^{\infty}(\lambda)^k x^{(k+1)(\alpha-\beta)}\, G_{p+k+1,q+k+1}^{m,n+k+1}\left[x\left|\begin{matrix}\vec{E}_{k+1}\\ \vec{F}_{k+1}\end{matrix}\right.\right] \tag{2.27}$$

We shall now analyze the convergence properties of the above series. To this aim we apply the comparison test. When $\kappa < \gamma_k = \gamma < \min[\vartheta, \vartheta_1] = \vartheta_1$ and $a^* > 0$ or $a^* = 0$, $\Delta\gamma + \mu < -1$, we obtain the following estimation for the modulus of the respective Meijer G-function (we apply Theorem 3.4 from [13]):

$$\left| G^{m,n+k+1}_{p+k+1,q+k+1}\left[x \middle| \begin{matrix} \vec{E}_{k+1} \\ \vec{F}_{k+1} \end{matrix} \right] \right| \leq A_{\gamma,k} x^{-\gamma} \tag{2.28}$$

where coefficients $A_{\gamma,k}$ are constants dependent on γ and k. Let us calculate these coefficients. According to the calculations enclosed in the proof of Theorem 3.3 from [13] it is a product of coefficients $A_{1,k}$ and $A_{2,k}$, given below

$$A_{1,k} = (2\pi)^{(q-p)/2} \exp\left(k\alpha - \mu - \Delta\left(\frac{1}{2} + \gamma \right) \right) \tag{2.29}$$

$$A_{2,k} = \frac{\pi^{m+n-q}}{\exp(-(n+k+1)\pi\tau)(\sinh(\pi\tau))^{n+k+1}} \tag{2.30}$$

where $\tau > 0$. Using formulas (2.28)-(2.30) in the estimation of solution (2.27) we obtain a majorizing series in the form of

$$g(x) = \sum_{k=0}^{\infty} |\lambda|^k A_{\gamma,k}\, x^{(k+1)(\alpha-\beta)-\gamma} \tag{2.31}$$

The above majorizing series is absolutely convergent for $x \in [0,b]$ using the d'Alembert test, provided eigenvalue λ obeys the inequality:

$$|\lambda| < \frac{\sin h(\pi\pi\tau}{b^{\alpha-\beta} \exp(\alpha + \pi\tau)} \tag{2.32}$$

Remark: *the above restriction can be relaxed using the fixed point theorem. We shall not discuss the details of the procedure in this paper leaving it for the next article. In the formulation of the proposition concerning the solution of equation* (2.1) *we shall omit assumption* (2.32).

Let us now check explicitly that series (2.27) solves equation (2.1). We denote the *k*-th element of the series as follows

$$\psi_k = (\lambda)^k x^{(k+1)(\alpha-\beta)} G^{m,n+k+1}_{p+k+1,q+k+1}\left[x \middle| \begin{matrix} \vec{E}_{k+1} \\ \vec{F}_{k+1} \end{matrix} \right] \tag{2.33}$$

and calculate the result of the action of operator $x^{\beta} D_{0+}^{\alpha}$ on this component:

$$x^{\beta} D_{0+}^{\alpha} \psi_k = (\lambda)^k x^{\beta} D_{0+}^{\alpha} x^{(k+1)(\alpha-\beta)} G_{p+k+1,q+k+1}^{m,n+k+1} \left[x \left| \begin{matrix} \vec{E}_{k+1} \\ \vec{F}_{k+1} \end{matrix} \right. \right] \tag{2.34}$$

From the theorem on the differentiation of Meijer G-functions (Theorem 2.8 in monograph [13]) it follows that

$$x^{\beta} D_{0+}^{\alpha} \psi_k = (\lambda)^k x^{k(\alpha-\beta)} G_{p+k+2,q+k+2}^{m,n+k+2} \left[x \left| \begin{matrix} (k+1)(\beta-\alpha); \vec{E}_{k+1} \\ \vec{F}_{k+1}; (k+1)\beta - k\alpha \end{matrix} \right. \right]$$

Function $G_{p+k+2,q+k+2}^{m,n+k+2}$ can be rewritten using the reduction property given in Appendix A so as to obtain the formulas:

$$x^{\beta} D_{0+}^{\alpha} \psi_k = (\lambda)^k x^{k(\alpha-\beta)} G_{p+k,q+k}^{m,n+k} \left[x \left| \begin{matrix} \vec{E}_k \\ \vec{F}_k \end{matrix} \right. \right] \tag{2.35}$$

valid for any $k \in \mathbf{N_0}$ with vectors $\vec{E}_0 = \vec{a}$ and $\vec{F}_0 = \vec{b}$.

We now apply the derived formulas for terms $x^{\beta} D_{0+}^{\alpha} \psi_k$ and obtain for series (2.27) the following result

$$\begin{aligned} x^{\beta} D_{0+}^{\alpha} f_s(x) &= \sum_{k=0}^{\infty} (\lambda)^k x^{\beta} D_{0+}^{\alpha} x^{(k+1)(\alpha-\beta)} G_{p+k+1,q+k+1}^{m,n+k+1} \left[x \left| \begin{matrix} \vec{E}_{k+1} \\ \vec{F}_{k+1} \end{matrix} \right. \right] = \\ &= \sum_{k=0}^{\infty} (\lambda)^k x^{k(\alpha-\beta)} G_{p+k,q+k}^{m,n+k} \left[x \left| \begin{matrix} \vec{E}_k \\ \vec{F}_k \end{matrix} \right. \right] \end{aligned} \tag{2.36}$$

which after substitution to equation (2.1) yields

$$\begin{aligned} (x^{\beta} D_{0+}^{\alpha} - \lambda) f_s(x) &= \sum_{k=0}^{\infty} (\lambda)^k x^{k(\alpha-\beta)} G_{p+k,q+k}^{m,n+k} \left[x \left| \begin{matrix} E_k \\ F_k \end{matrix} \right. \right] + \\ - \sum_{k=0}^{\infty} \lambda^{k+1} x^{(k+1)(\alpha-\beta)} G_{p+k+1,q+k+1}^{m,n+k+1} \left[x \left| \begin{matrix} E_{k+1} \\ F_{k+1} \end{matrix} \right. \right] &= G_{p,q}^{m,n} \left[x \left| \begin{matrix} \vec{a} \\ \vec{b} \end{matrix} \right. \right] \end{aligned} \tag{2.37}$$

From the above considerations the proposition describing the general solution of equation (2.1) follows.

Proposition 2.2

Let $\alpha \in (n-1, n), \alpha > \frac{1}{2}$ *and* $\{\alpha\} - \beta > 0$. *Let* $G^{m,n}_{p,q}$ *be an arbitrary Meijer G-function determined at least for* $x \in (0,b]$ *by real vectors* $\vec{a}$ *and* $\vec{b}$ *fulfilling the assumptions of Theorem A.2 and condition* $\kappa < \vartheta_1$, *where* κ *and* ϑ_1 *are given by formulas (2.10, 2.17). The differential equation of order* α:

$$(x^{\beta} D^{\alpha}_{0+} - \lambda) f(x) = G^{m,n}_{p,q}(x)$$

has in interval $[0, b]$ *a general solution in the form of the following sum*

$$f(x) = \sum_{l=1}^{n} c_l f_0^l(x) + f_s(x)$$

where component solutions f_0^l, f_s *are given by the series*

$$f_0^l(x) = b^{\alpha-l} \sum_{k=0}^{\infty} \left(\lambda\, b^{\alpha-\beta}\right)^k G^{1,\;k}_{k+1,k+1}\left[\frac{x}{b}\,\middle|\,\begin{matrix}\vec{A}_{k,l}\\ \vec{B}_{k,l}\end{matrix}\right]$$

$$f_s(x) = \sum_{k=0}^{\infty} (\lambda)^k x^{(k+1)(\alpha-\beta)} G^{m,n+k+1}_{p+k+1,q+k+1}\left[x\,\middle|\,\begin{matrix}\vec{E}_{k+1}\\ \vec{F}_{k+1}\end{matrix}\right]$$

with vectors $\vec{A}_{k,l}, \vec{B}_{k,l}, \vec{E}_{k+1}, \vec{F}_{k+1}$ *given by formulas*

$$\begin{aligned}
\vec{A}_{k,l} &= \left[\alpha \vec{e}_k + (\alpha-\beta)\vec{j}_k;\, k(\alpha-\beta)+\alpha-l+1\right]\\
\vec{B}_{k,l} &= \left[k(\alpha-\beta)+\alpha-l;\, (\alpha-\beta)\vec{j}_k\right]\\
\vec{E}_{k+1} &= \left[\beta \vec{e}_{k+1} + (\beta-\alpha)\vec{j}_{k+1};\, \vec{a}\right]\\
\vec{F}_{k+1} &= [\vec{b}; (\beta-\alpha)\vec{e}_{k+1} + (\beta-\alpha)\vec{j}_{k+1}]\\
\vec{e}_k &= [1, ..., 1] \in \mathbf{R}^k \quad \vec{j}_k = [0, 1, ..., k-1] \in \mathbf{R}^k
\end{aligned}$$

and arbitrary real coefficients $c_l \in \mathbf{R}$.

The above proposition can be easily extended to the case where the non-homogeneous term on the right-hand side of equation (2.1) is a linear combination of the Meijer G-functions obeying the corresponding assumptions.

Corollary 2.3

Let $\alpha \in (n-1, n)$, $\alpha > \frac{1}{2}$ *and* $\{\alpha\} - \beta > 0$. *Let* $G^{m_r, n_r}_{p_r, q_r}$ *be arbitrary Meijer G-functions determined at least for* $x \in (0,b]$ *by real vectors* $\vec{a}_r$ *and* $\vec{b}_r$, *fulfilling the assumptions of Theorem A.2 and conditions* $\kappa^r < \vartheta_1^r$ *for* $r = 1,...,R$, *where* κ^r *and* ϑ_1^r *are given by formulas* (2.10,2.17). *The differential equation of order α:*

$$(x^\beta D^\alpha_{0+} - \lambda) f(x) = \sum_{r=1}^{R} C_r G^{m_r, n_r}_{p_r, q_r}(x)$$

has in interval $[0, b]$ *a general solution in the form of the following sum*

$$f(x) = \sum_{l=1}^{n} c_l f_0^l(x) + \sum_{r=1}^{R} C_r f_s^r(x)$$

where component solutions f_0^l, f_s^r *are given as series*

$$f_0^l(x) = b^{\alpha - l} \sum_{k=0}^{\infty} \left(\lambda\, b^{\alpha-\beta}\right)^k G^{1,\ k}_{k+1, k+1}\left[\frac{x}{b} \,\middle|\, \begin{matrix} \vec{A}_{k,l} \\ \vec{B}_{k,l} \end{matrix}\right]$$

$$f_s^r(x) = \sum_{k=0}^{\infty} (\lambda)^k x^{(k+1)(\alpha-\beta)} G^{m_r, n_r + k+1}_{p_r + k+1, q_r + k+1}\left[x \,\middle|\, \begin{matrix} \vec{E}_{r,k+1} \\ \vec{F}_{r,k+1} \end{matrix}\right]$$

with vectors described in detail by Proposition 2.2.

3. Examples

We shall study in detail the two example equations solved in Proposition 2.2. We chose such values of parameter β to which the reduction properties of Meijer functions can be applied.

3.1. Example: case $\boldsymbol{\beta = 0}$. Let us solve equation (2.1) when $\beta = 0$:

$$(D^\alpha_{0+} - \lambda) f(x) = G^{m,n}_{p,q}(x) \quad x \in [0,b] \tag{3.1}$$

Assumptions of Proposition 2.2 read $\alpha > \frac{1}{2}$ and $\kappa < \vartheta_1$. Following the results of paper [10], we obtain the general solution of the homogeneous part of equation (3.1) in the form of a linear combination of Mittag-Leffler functions [7]:

$$f_0(x) = \sum_{l=1}^{n} c_l f_0^l(x) \tag{3.2}$$

$$f_0^l(x) = x^{\alpha - l} E_{\alpha,\, \alpha - l + 1}(\lambda x^{\alpha}) \tag{3.3}$$

From Proposition 2.2 and Property A.1 we infer that the particular solution of the non-homogeneous part of equation (3.1) is given as

$$f_s(x) = \sum_{k=0}^{\infty} (\lambda)^k x^{\alpha(k+1)} G_{p+1,q+1}^{m,n+1}\left[x \,\middle|\, \begin{matrix} \vec{E}_{k+1} \\ \vec{F}_{k+1} \end{matrix} \right] \tag{3.4}$$

where vectors $\vec{E}_{k+1}$, $\vec{F}_{k+1}$ belong to the $\mathbf{R}^{p+1}$ and $\mathbf{R}^{q+1}$ spaces respectively and look as follows

$$\vec{E}_{k+1} = [0; \vec{a}] \qquad \vec{F}_{k+1} = [\vec{b}; (k+1)\alpha]$$

Concluding, the general solution of equation (3.1) is constructed using the f_s - solution and solutions f_0^l as the following sum

$$f(x) = \sum_{l=1}^{n} c_l f_0^l(x) + f_s(x) \tag{3.5}$$

3.2. Example: case *β* = *α*/2. In the next example, we study case $\beta = \alpha/2$. Let us observe that according to assumption $\{\alpha\} - \beta > 0$, it requires $\alpha \in (0,1)$. Thus, equation (2.1) becomes an equation of fractional order $\alpha \in \left(\frac{1}{2}, 1\right)$ and we assume that the conditions of Proposition 2.2 regarding a Meijer G-function on the right-hand side are fulfilled:

$$(x^{\alpha/2} D_{0+}^{\alpha} - \lambda) f(x) = G_{p,q}^{m,n}(x) \quad x \in [0,b] \tag{3.6}$$

The solution described in Proposition 2.2 includes the general solution of the homogeneous part:

$$f_0(x) = c_1 f_0^1(x) \tag{3.7}$$

where the component series was derived in paper [10]

$$f_0^1(x) = b^{\alpha-1} \sum_{k=0}^{\infty} \left(\lambda\, b^{\alpha/2}\right)^k G_{3,3}^{1,2}\left[\frac{x}{b} \,\middle|\, \begin{matrix} \vec{A}_{k,1} \\ \vec{B}_{k,1} \end{matrix} \right] \tag{3.8}$$

with vectors $A_{k,1}$, $B_{k,1}$ given by formulas

$$\vec{A}_{k,1} = \left[\frac{\alpha}{2}k; \frac{\alpha}{2}(k+1); \frac{\alpha}{2}(k+2)\right] \quad \vec{B}_{k,1} = \left[\frac{\alpha}{2}(k+2)-1; 0; \frac{\alpha}{2}\right]$$

The second part of the solution is provided by Proposition 2.2 as the following series

$$f_s(x) = \sum_{k=0}^{\infty} (\lambda)^k x^{\alpha(k+1)/2} G^{m,n+2}_{p+2,q+2}\left[x \,\middle|\, \begin{matrix} \vec{E}_{k+1} \\ \vec{F}_{k+1} \end{matrix}\right] \tag{3.9}$$

with vectors $\vec{E}_{k+1}$, $\vec{F}_{k+1}$ in the form of

$$\vec{E}_{k+1} = \left[\frac{\alpha}{2}; 0; \vec{a}\right] \quad \vec{F}_{k+1} = \left[\vec{b}; -\frac{\alpha}{2}k; -\frac{\alpha}{2}(k+1)\right]$$

Concluding, the solution of equation (3.6) is the sum of the above series (3.8) and (3.9)

$$f(x) = c_1 f_0^1(x) + f_s(x) \tag{3.10}$$

4. Final remarks

We have introduced a method of solving a basic non-homogeneous fractional differential equation. It includes application of the integration properties of Meijer G-functions. A class of equations including as a non-homogeneous term an arbitrary linear combination of the Meijer functions, is solved applying the discussed procedure. Careful analysis shows that the solution technique can be extended to equations with Fox functions in the non-homogeneous term. Our further goal is to apply the results of the considered equations to a class of sequential non-homogeneous fractional differential equations.

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Appendix A

Here we recall the basic properties of Fox and Meijer functions, which we applied in the procedure of solving the fractional differential equations [13].

Property A.1

(1) If $b_q = a_1$, then the following reduction formula holds

$$G^{m,n}_{p,q}\left[z\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right] = G^{m,n-1}_{p-1,q-1}\left[z\left|\begin{matrix}a_2,\dots,a_p\\ b_1,\dots,b_{q-1}\end{matrix}\right.\right] \tag{A.1}$$

provided $n \geq 1$ and $q > m$.

(2) If $a_p = b_1$, then the reduction formula holds

$$G^{m,n}_{p,q}\left[z\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right] = G^{m-1,n}_{p-1,q-1}\left[z\left|\begin{matrix}a_1,\dots,a_{p-1}\\ b_2,\dots,b_q\end{matrix}\right.\right] \tag{A.2}$$

provided $m \geq 1$ and $p > n$.

Theorem A.2

Let $\alpha \in \mathbf{C}$, where $Re(\alpha) > 0$, $\omega \in \mathbf{C}$ and $\sigma > 0$. If the conditions hold

$$\sigma \min_{1\leq j\leq m}\left[\frac{Re(b_j)}{\beta_j}\right] + Re(\omega) > -1 \tag{A.3}$$

provided $a^* > 0$ or $a^* = 0$, $\Delta \geq 0$, $\mathrm{Re}(\mu) < -1$ or

$$\sigma \min_{1\leq j\leq m}\left[\frac{Re(b_j)}{\beta_j}, \frac{Re(\mu)+\frac{1}{2}}{\Delta}\right] + Re(\omega) > -1 \tag{A.4}$$

provided $a^* = 0, \Delta < 0$, where $Re(\mu) < -1$,
then the left-sided fractional integral of the Fox H-function exists and is given by the formula below:

$$\left(I_{0+}^{\alpha} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]\right)(x) = x^{\omega+\alpha} H_{p+1,q+1}^{m,n+1}\left[x^{\sigma}\left|\begin{matrix}(-\omega,\sigma),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(-\omega-\alpha,\sigma)\end{matrix}\right.\right] \tag{A.5}$$