INTEGRATION ON HYPERCIRCLES IN $\boldsymbol{R}^n$

***Grzegorz Biernat*[1], *Magdalena Ligus*[2], *Jarosław Siedlecki*[1]**

[1] *Institute of Mathematics, Czestochowa University of Technology, Poland*
[2] *Faculty of Mechanical Engineering and Computer Science*
Czestochowa University of Technology, Poland

Abstract. This paper presents the formulas parametrizing hypercircles (the intersections of hyperplanes with the sphere in $\boldsymbol{R}^n$, $n \geq 3$). A hypothesis concerning the integral on hypercircle C_{n-2} of the $n-2$ - canonical form ω_{n-2} is proposed.

1. Parametrization of hypercircles

A plane circle with the centre at 0 and radius r, has the given parametrization

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad (0 \leq t \leq 2\pi) \tag{1}$$

How about the case of a circle in 3-dimensional space? It is easy to imagine, but how to describe it? Certainly, as an intersection of a plane with a sphere. If we restrict our considerations to circles with the centre at 0, then the system of equation is given as

$$\begin{matrix} Ax + By + Cz = 0 & (plane, A^2 + B^2 + C^2 > 0) \\ x^2 + y^2 + z^2 = r^2 & (sphere, r > 0) \end{matrix} \tag{2}$$

The parametrizations, mentioned above, are difficult to apply in integration around circle. Thus, parametric description is used. The construction follows.

1° $A = B = 0$. Then, the circle is described by parametrization

$$\begin{cases} x = r\cos t \\ y = r\sin t \\ z = 0 \end{cases} \qquad (0 \leq t \leq 2\pi) \tag{3}$$

2° $A^2 + B^2 > 0$. In this case, we construct on the plane $Ax + By + Cz = 0$ a set of orthogonal vectors $[B, -A, 0]$, $\left[AB, AC, -A^2 - B^2\right]$ and $[A, B, C]$. We obtain the following orthonormal basis in the space $\boldsymbol{R}^3$:

$$v_1 = \left[\frac{B}{\sqrt{A^2+B^2}}, \frac{-A}{\sqrt{A^2+B^2}}, 0 \right] \tag{4}$$

$$v_2 = \left[\frac{AC}{\sqrt{A^2+B^2} \cdot \sqrt{A^2+B^2+C^2}}, \frac{BC}{\sqrt{A^2+B^2} \cdot \sqrt{A^2+B^2+C^2}}, \frac{-A^2-B^2}{\sqrt{A^2+B^2} \cdot \sqrt{A^2+B^2+C^2}} \right] \tag{5}$$

$$v_3 = \left[\frac{A}{\sqrt{A^2+B^2+C^2}}, \frac{B}{\sqrt{A^2+B^2+C^2}}, \frac{C}{\sqrt{A^2+B^2+C^2}} \right] \tag{6}$$

The parametrized circle in the above coordinate system has a typical description given by formula (3).

We obtain the system of equations

$$\begin{bmatrix} \frac{B}{\sqrt{A^2+B^2}} & \frac{-A}{\sqrt{A^2+B^2}} & 0 \\ \frac{AC}{\sqrt{A^2+B^2}\sqrt{A^2+B^2+C^2}} & \frac{BC}{\sqrt{A^2+B^2}\sqrt{A^2+B^2+C^2}} & \frac{-A^2-B^2}{\sqrt{A^2+B^2}\sqrt{A^2+B^2+C^2}} \\ \frac{A}{\sqrt{A^2+B^2+C^2}} & \frac{B}{\sqrt{A^2+B^2+C^2}} & \frac{C}{\sqrt{A^2+B^2+C^2}} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos t \\ r\sin t \\ 0 \end{bmatrix} \tag{7}$$

where the determinant of the coefficient matrix is equal to 1.

The solution of the system looks as follows

$$x = r\left(\frac{B}{\sqrt{A^2+B^2}}\cos t + \frac{AC}{\sqrt{A^2+B^2}\sqrt{A^2+B^2+C^2}}\sin t \right) \tag{8}$$

$$y = r\left(-\frac{A}{\sqrt{A^2+B^2}}\cos t + \frac{BC}{\sqrt{A^2+B^2}\sqrt{A^2+B^2+C^2}}\sin t \right) \tag{9}$$

$$z = -r\sqrt{\frac{A^2+B^2}{A^2+B^2+C^2}}\sin t \tag{10}$$

where $0 \le t \le 2\pi$.

Thereafter, we consider the case, where hyperplane H_n is defined by

$$A_1x_1 + A_2x_2 + ... + A_nx_n = 0,\ A_1^2 + A_2^2 + ... + A_n^2 > 0,\ n \geq 3 \tag{11}$$

and sphere S_n intersects $\boldsymbol{R^n}$

$$x_1^2 + x_2^2 + ... + x_n^2 = r^2,\ r > 0 \tag{12}$$

Let us refer to these as intersections hypercircles C_{n-2} (for $n = 3$ we obtain a circle). Obviously, hypercircles C_{n-2} have the dimension $n-2$. Next, the following orthogonal versors can be constructed (the first $n-1$ on hyperplane H_n).

Let us assume that $A_1 \neq 0$.

The *first versor* is defined as

$$v_1 = \left[\frac{A_2}{\sqrt{A_1^2 + A_2^2}}, \frac{-A_1}{\sqrt{A_1^2 + A_2^2}}, 0, ..., 0 \right] \tag{13}$$

Certainly, v_1 is orthogonal to the vector $[A_1, A_2, ..., A_n]$ and $v_1 \subset H_n$.
We determine versor v_2 by adopting the following formula

$$v_2 = \left[\frac{A_1A_3}{\sqrt{(A_1^2 + A_2^2)(A_1^2 + A_2^2 + A_3^2)}}, \frac{A_2A_3}{\sqrt{(A_1^2 + A_2^2)(A_1^2 + A_2^2 + A_3^2)}}, \right. \\ \left. \frac{-A_1^2 - A_2^2}{\sqrt{(A_1^2 + A_2^2)(A_1^2 + A_2^2 + A_3^2)}}, 0, ..., 0 \right] \tag{14}$$

versor v_2 is orthogonal to vectors $[A_1, A_2, ..., A_n]$, v_1 and $v_2 \subset H_n$.

The third versor is defined as follows

$$v_3 = \left[\frac{A_1A_4}{\sqrt{\left(A_1^2 + A_2^2 + A_3^2\right)\left(A_1^2 + A_2^2 + A_3^2 + A_4^2\right)}}, \frac{A_2A_4}{\sqrt{\left(A_1^2 + A_2^2 + A_3^2\right)\left(A_1^2 + A_2^2 + A_3^2 + A_4^2\right)}}, \right. \\ \left. \frac{A_3A_4}{\sqrt{\left(A_1^2 + A_2^2 + A_3^2\right)\left(A_1^2 + A_2^2 + A_3^2 + A_4^2\right)}}, \frac{-A_1^2 - A_2^2 - A_3^2}{\sqrt{\left(A_1^2 + A_2^2 + A_3^2\right)\left(A_1^2 + A_2^2 + A_3^2 + A_4^2\right)}}, 0, ..., 0 \right] \tag{15}$$

Versor v_3 is orthogonal to vectors $[A_1, A_2, ..., A_n]$, v_1, v_2 and $v_3 \subset H_n$.

Generally, we obtain (applying the vector product [1, 5])

$$v_k = \left[\frac{A_1 A_{k+1}}{\sqrt{\left(A_1^2+\ldots+A_k^2\right)\left(A_1^2+\ldots+A_{k+1}^2\right)}},\ldots,\frac{A_k A_{k+1}}{\sqrt{\left(A_1^2+\ldots+A_k^2\right)\left(A_1^2+\ldots+A_{k+1}^2\right)}} \right.$$
$$\left. ,\frac{-A_1^2-\ldots-A_k^2}{\sqrt{\left(A_1^2+\ldots+A_k^2\right)\left(A_1^2+\ldots+A_{k+1}^2\right)}},0,\ldots,0 \right] \tag{16}$$

for $1 \le k \le n-1$ (if $k = n-1$, then the last component is not 0).
Evidently versor v_k is orthogonal to the vectors $\left[A_1, A_2,\ldots,A_n\right], v_1,\ldots,v_{k-1}$ and $v_k \subset H_n$.

The last versor

$$v_n = \left[\frac{A_1}{\sqrt{A_1^2+A_2^2+\ldots+A_n^2}},\frac{A_2}{\sqrt{A_1^2+A_2^2+\ldots+A_n^2}},\ldots,\frac{A_n}{\sqrt{A_1^2+A_2^2+\ldots+A_n^2}} \right] \tag{17}$$

is orthogonal to hyperplane H_n.

Let us define matrix U as follows

$$U := \left[v_1, v_2, v_3,\ldots,v_n\right]^T \tag{18}$$

Next, we recall sphere parameterization [1, 2, 4]

$$\begin{gathered} x_1 = r\cos t_1 \cos t_2 \ldots \cos t_{n-1} \\ x_2 = r\sin t_1 \cos t_2 \ldots \cos t_{n-1} \\ x_1 = r\sin t_2 \cos t_3 \ldots \cos t_{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1} = r\sin t_{n-2} \cos t_{n-1} \\ x_n = r\sin t_{n-1} \end{gathered} \tag{19}$$

where $0 \le t_1 \le 2\pi$, $-\frac{\pi}{2} \le t_j \le \frac{\pi}{2}$, $2 \le j \le n-1$.

We calculate the hypercircle parametrization by solving the following system of equations

$$U \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} r\cos t_1 \cos t_2 ... \cos t_{n-2} \\ r\sin t_1 \cos t_2 ... \cos t_{n-2} \\ r\sin t_1 \cos t_2 ... \cos t_{n-2} \\ \dots\dots\dots\dots \\ r\sin t_{n-2} \\ 0 \end{bmatrix} \tag{20}$$

Because matrix U is orthogonal, the solution is given by vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix} = U^T \cdot \begin{bmatrix} r\cos t_1 \cos t_2 ... \cos t_{n-2} \\ r\sin t_1 \cos t_2 ... \cos t_{n-2} \\ r\sin t_1 \cos t_2 ... \cos t_{n-2} \\ \dots\dots\dots\dots \\ r\sin t_{n-2} \\ 0 \end{bmatrix} \tag{21}$$

2. Integration around hypercircles

We calculate some integrals around circle $\boldsymbol{C_1}$ in $\boldsymbol{R^3}$ and hypercircle $\boldsymbol{C_2}$ in $\boldsymbol{R^4}$ [6]. Using the results from the previous section (three dimensional case of intersection of a sphere with the equation $x^2 + y^2 + z^2 = r^2$ of the plane $Ax + By + Cz = 0$ where $A^2 + B^2 + C^2 > 0$) we can calculate the following integral:

$$\int_{C_1} \frac{Bz - Cy}{x^2 + y^2 + z^2} dx + \frac{Cx - Az}{x^2 + y^2 + z^2} dy + \frac{Ay - Bx}{x^2 + y^2 + z^2} dz \tag{22}$$

The above integral is given by the formula

$$\int_{C_1} \frac{Bz - Cy}{x^2 + y^2 + z^2} dx + \frac{Cx - Az}{x^2 + y^2 + z^2} dy + \frac{Ay - Bx}{x^2 + y^2 + z^2} dz =$$

$$\int_0^{2\pi} \frac{1}{x^2 + y^2 + z^2} \begin{vmatrix} A & x & \frac{dx}{dt} \\ B & y & \frac{dy}{dt} \\ C & z & \frac{dz}{dt} \end{vmatrix} dt = \int_0^{2\pi} \frac{1}{x^2 + y^2 + z^2} \left| U^T \right| \cdot \begin{vmatrix} A' & r\cos t & -r\sin t \\ B' & r\sin t & r\cos t \\ C' & 0 & 0 \end{vmatrix} dt = \tag{23}$$

$$\int_0^{2\pi} \frac{1}{r^2} \cdot C' \cdot r^2 \cdot \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix} dt = \int_0^{2\pi} C' \, dt = 2\pi \sqrt{A^2 + B^2 + C^2}$$

where

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = U \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} \tag{24}$$

Thus, integral (22) is equal to the length of a circle with the radius $\sqrt{A^2+B^2+C^2}$ [6]. We can now consider and calculate the following integrate on hypercircle C_2 in $\boldsymbol{R^4}$:

$$\int_{C_2} \frac{\begin{vmatrix} A_4 & A_3 \\ x_4 & x_3 \end{vmatrix}}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{3}{2}}} dx_1 \wedge dx_2 - \frac{\begin{vmatrix} A_4 & A_2 \\ x_4 & x_2 \end{vmatrix}}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{3}{2}}} dx_1 \wedge dx_3 +$$

$$\frac{\begin{vmatrix} A_3 & A_2 \\ x_3 & x_2 \end{vmatrix}}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{3}{2}}} dx_1 \wedge dx_4 + \frac{\begin{vmatrix} A_4 & A_1 \\ x_4 & x_1 \end{vmatrix}}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{3}{2}}} dx_2 \wedge dx_3 -$$

$$\frac{\begin{vmatrix} A_3 & A_1 \\ x_3 & x_1 \end{vmatrix}}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{3}{2}}} dx_2 \wedge dx_4 - \frac{\begin{vmatrix} A_2 & A_1 \\ x_2 & x_1 \end{vmatrix}}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{3}{2}}} dx_3 \wedge dx_4 =$$

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{3}{2}}} \cdot \begin{vmatrix} A_1 & x_1 & \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ A_2 & x_2 & \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \\ A_3 & x_3 & \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \\ A_4 & x_4 & \frac{\partial x_4}{\partial t_1} & \frac{\partial x_4}{\partial t_2} \end{vmatrix} dt_2 dt_1 = \tag{25}$$

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\left(r^2\right)^{\frac{3}{2}}} \left|U^T\right| \cdot \begin{vmatrix} A'_1 & r\cos t_1 \cos t_2 & -r\sin t_1 \cos t_2 & -r\cos t_1 \sin t_2 \\ A'_2 & r\sin t_1 \cos t_2 & r\cos t_1 \cos t_2 & -r\sin t_1 \sin t_2 \\ A'_3 & r\sin t_2 & 0 & r\cos t_2 \\ A'_4 & 0 & 0 & 0 \end{vmatrix} dt_2 dt_1 =$$

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{r^3} \cdot A'_4 \cdot r^3 \cdot \cos t_2 dt_2 dt_1 = A'_4 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t_2 dt_2 dt_1 = A'_4 \int_0^{2\pi} dt_1 \sin t_2 \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$2A'_4 \cdot 2\pi = 4\pi \cdot A'_4 = 4\pi\sqrt{A_1^2+A_2^2+A_3^2+A_4^2} \ ,$$

where

$$\begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{bmatrix} = U \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \tag{26}$$

We observe that integral (25) is equal to the surface of a sphere with the radius $\sqrt{A_1^2 + A_2^2 + A_3^2 + A_4^2}$ [6]. The above results suggest the following hypothesis.

3. Hypothesis

Integral around hypercircle C_{n-2} of the $n-2$ - canonical form [3]

$$\omega_{n-2} = \frac{(-1)^{n-1}}{\left(x_1^2 + \ldots + x_n^2\right)^{\frac{n-1}{2}}} \sum_{1 \le i_1 < i_2 < \ldots < i_{n-2} \le n} \; \operatorname{sgn}_{\substack{1 \le j_1 < j_2 \le n \\ j_1, j_2 \ne i_1, i_2, \ldots, i_{n-2}}} (i_1, \ldots, i_{n-2}, j_1, j_2) \times \det \begin{bmatrix} A_{j_1} & A_{j_2} \\ x_{j_1} & x_{j_2} \end{bmatrix} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_{n-2}} \tag{27}$$

is equal to

$$\int_{C_{n-2}} \omega_{n-2} = \left(\begin{matrix} \textit{surface of} \\ (n-2)-\textit{dimensional unit sphere } [1] \end{matrix} \right) \sqrt{A_1^2 + A_2^2 + \ldots + A_n^2} \tag{28}$$

Parametrizations and some integrals around hyperspheres will be presented in the next paper.

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