THE POLYNOMIAL TENSOR INTERPOLATION. ARITHMETICAL CASE

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Abstract. In this paper the tensor interpolation by polynomials of several variables is considered. The effective formulas for polynomial coefficients for arithmetical case were obtained.

Introduction

The formulas of tensor interpolation by polynomials of several variables are unknow in the interpolation methods [1]. Using the Kronecker tensor product of matrices [2, 3] the polynomial tensor interpolation formula was given in the previous articles [4, 5]. In this paper we considere the arithmetical case of the nodes matrix.

The Polynomial Arithmetical Tensor Interpolation

The coefficients matrix $[a_{j_1 \dots j_k}]\left[w_{i_1 \dots i_k}\right]$ of the polynomial arithmetic tensor interpolation

$$W(X_1, \dots, X_k) = \sum_{0 \le j_1 \le p_1, \dots, 0 \le j_k \le p_k} a_{j_1 \dots j_k} X_1^{j_1} \dots X_k^{j_k}$$

are unknow.

The results matrix $\left[(w_1)_{i_1} \dots (w_k)_{i_k}\right] = \left[w_{i_1 \dots i_k}\right]$ and the nodes matrix $\left[(X_1)_{i_1} \times \dots \times (X_k)_{i_k}\right] = \left[X_{1i_1} \dots X_{ki_k}\right]$ are know and

$$\begin{aligned}
X_{1i_1} &= X_{10} + i_1 \Delta_1, && 0 \le i_1 \le p_1 \\
X_{2i_2} &= X_{20} + i_2 \Delta_2, && 0 \le i_2 \le p_2 \\
&\dots\dots\dots\dots\dots\dots \\
X_{ki_k} &= X_{k0} + i_k \Delta_k, && 0 \le i_k \le p_k
\end{aligned}$$

where Δ_l is a common difference of X_{li_l} sequence.

Fact 1. *For the arithmetical sequence* $X_i = X_0 + i\Delta$ $(i = 0,1,\ldots,p)$ *we have* [5]

$$\begin{aligned}\Pi_i &= (X_p - X_i)\cdot\ldots\cdot(X_{i+1} - X_i)\cdot(X_i - X_{i-1})\cdot\ldots\cdot(X_i - X_0) = \\ &(p-i)\Delta\cdot\ldots\cdot\Delta\cdot\Delta\cdot 2\Delta\cdot\ldots\cdot i\Delta = \\ &(p-i)!\Delta^{p-i} i!\Delta^i = (p-i)!i!\Delta^p\end{aligned}$$

Lemma 1. *For integers* $1 \le q \le l \le p$ *we have*

$$\sum\nolimits_{1\le\kappa_1<\kappa_2,\ldots,<\kappa_l\le p} \tau_q(\kappa_1,\ldots,\kappa_l) = \binom{p-q}{l-q}\tau_q(1,2,\ldots,p)$$

where τ_q is the symmetric polynomial of q - order.

Proof. For the numbers $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_q \le p$ from sequence $(\kappa_1, \kappa_2, \ldots, \kappa_l)$ we have *p-l* remaining values disposed to *q-l* places. So, each component $\alpha_1\alpha_2\ldots\alpha_q$ of symmetric polynomial $\tau_q(1,2,\ldots,p)$ is repeated in the left hand sum $\binom{p-q}{l-q}$ times.

Lemma 2. *For integers* $1 \le q \le p$ *we have*

$$\tau_q(1,2,\ldots,\hat{i},\ldots,p) = \tau_q(1,2,\ldots,p) - i\tau_{q-1}(1,2,\ldots,\hat{i},\ldots,p)$$

where the symbol $\hat{i}$ means omitting the variable *i*.

Corollary 1. *For integers* $1 \le q \le p$ *we obtain*

$$\tau_q(1,2,\ldots,\hat{i},\ldots,p) = \tau_q(1,2,\ldots,p) - i\tau_{q-1}(1,2,\ldots,p) +$$

$$i^2\tau_{q-2}(1,2,\ldots,p) + \ldots + (-1)^q i^q \tau_0$$

Fact 2. *For aritmetical sequence* $X_i = X_0 + i\Delta$ $(i = 0,1,\ldots.,p)$ *we have*

$$\tau_l(X_0, X_1, ..., \hat{X}_i, ..., X_p) = \sum_{q=0}^{l} \binom{p-q}{l-q} \left(\sum_{s=0}^{q} (-1)^s i^s \tau_{q-s}(1,2,\dots,p) \right) \Delta^q X_0^{l-q}$$

In this formula we assumme that $0^0 = 1$.

Proof. In fact, according to lemma 1 and corollary 1 we obtain

$$\tau_l(X_0, X_1, \dots, \hat{X}_i, \dots, X_p) = \sum_{\substack{0 \le k_1 < k_2, \dots, < k_l \le p \\ k_j \ne i, j=1,2,\dots,p}} X_{k_1} \dots X_{k_l} =$$

$$\sum_{\substack{0 \le k_1 < k_2, \dots, < k_l \le p \\ k_j \ne i, j=1,2,\dots p}} (X_0 + k_1 \Delta) \dots (X_0 + k_l \Delta) =$$

$$\sum_{\substack{0 \le k_1 < k_2, \dots, < k_l \le p \\ k_j \ne i, j=1,2,\dots,p}} (X_0^l + \tau_1(k_1, \dots k_l) \Delta X_0^{l-1} +$$

$$\tau_2(k_1, \dots, k_l) \Delta^2 X_0^{l-2} + \dots + \tau_l(k_1, \dots, k_l) \Delta X_0^{l-l}) =$$

$$\binom{p}{l} X_0^l + \left(\sum_{\substack{0 \le k_1 < k_2, \dots, < k_l \le p \\ k_j \ne i, j=1,2,\dots,p}} \tau_1(k_1, \dots, k_l) \right) \Delta X_0^{l-1} +$$

$$\left(\sum_{\substack{0 \le k_1 < k_2, \dots, < k_l \le p \\ k_j \ne i, j=1,2,\dots p}} \tau_2(k_1, \dots, k_l) \right) \Delta^2 X_0^{l-2} + \dots +$$

$$\left(\sum_{\substack{0 \le k_1 < k_2, \dots, < k_l \le p \\ k_j \ne i, j=1,2,\dots,p}} \tau_l(k_1, \dots, k_l) \right) \Delta^l =$$

$$\binom{p}{l} X_0^l + \binom{p-1}{l-1} \tau_1(1, 2, \dots, \hat{i}, \dots, p) \Delta X_0^{l-1} +$$

$$\binom{p-2}{l-2} \tau_2(1, 2, \dots, \hat{i}, \dots, p) \Delta^2 X_0^{l-2} + \dots + \tau_l(1, 2, \dots, \hat{i}, \dots, p) \Delta^l =$$

$$\binom{p}{l} X_0^l + \sum_{q=1}^{l} \binom{p-q}{l-q} \left(\sum_{s=0}^{q} (-1)^s i^s \tau_{q-s}(1, 2, \dots, p) \right) \Delta^q X_0^{l-q}$$

According to above facts we obtain

Corollary 2. *In the formulas of polynomial coefficients of arithmetical tensor interpolation* [5] *we have*

$$\tau_{p_1-j_1}(X_{10},\dots,\hat{X}_{1i_k},\dots,X_{1p_1})=\sum_{q_1=0}^{p_1-j_1}\binom{p_1-q_1}{p_1-j_1-q_1}\sum_{s_1=0}^{q_1}(-1)^{s_1}i_1^{s_1}\tau_{q_1-s_1}(1,2,\dots,p_1)\Delta^{q_1}X_{10}^{p_1-j_1-q_1}$$

....

$$\tau_{p_k-jk}(X_{k0},\dots,\hat{X}_{ki_k},\dots,X_{kp_k})=\sum_{q_k=0}^{p_k-j_k}\binom{p_k-q_k}{p_k-j_k-q_k}\sum_{s_k=0}^{q_k}(-1)^{s_k}i_k^{s_k}\tau_{q_k-s_k}(1,2,\dots,p_k)\Delta^{q_k}X_{k0}^{p_k-j_k-q_k}$$

and

$$\Pi_{1i_1}=(p_1-i_1)!i_1!\Delta_1^{p_1}$$

....

$$\Pi_{ki_k}=(p_k-i_k)!i_k!\Delta_k^{p_k}$$

so

$$a_{j_1\dots j_k}=\frac{(-1)^{j_1+\dots+j_k}}{\Delta_1^{p_1}\dots\Delta_k^{p_k}}\Bigg(\sum\nolimits_{0\le i_1\le p_1,\dots,0\le i_k\le p_k}(-1)^{i_1+\dots+i_k}w_{i_1\dots i_k}$$

$$\frac{\sum_{q_1=0}^{p_1-j_1}\binom{p_1-q_1}{p_1-j_1-q_1}\sum_{s_1=0}^{q_1}(-1)^{s_1}i_1^{s_1}\tau_{q_1-s_1}(1,2,\dots,p_1)\Delta^{q_1}X_{10}^{p_1-j_1-q_1}}{(p_1-i_1)!i_1!}\dots$$

$$\frac{\sum_{q_k=0}^{p_k-j_k}\binom{p_k-q_k}{p_k-j_k-q_k}\sum_{s_k=0}^{q_k}(-1)^{s_k}i_k^{s_k}\tau_{q_k-s_k}(1,2,\dots,p_k)\Delta^{q_k}X_{k0}^{p_k-j_k-q_k}}{(p_k-i_k)!i_k!}\Bigg)$$

In above formulas we assumme that $0^0=1$.

Conclusion

In this article the effective formulas for the polynomial coefficients of arithmetical tensor interpolation were obtained. Only the values of symmetric polynomials for natural numbers $\tau_q(1,2,\dots,p)$ are necessary.

References

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