FINITE DIFFERENCE METHOD IN THE FOURIER EQUATION WITH NEWTON'S BOUNDARY CONDITIONS DIRECT FORMULAS

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Abstract. In the paper we give the direct FDM formulas for the solutions of the Fourier equation with the Newton boundary condition in the (x,t) case.

1. Formulation of the problem

In limited spatial solid (centres) approximate to the one-dimensional case the temperature distribution $T(x,t)$ with the Newton boundary conditions and determined initial conditions (without the internal heat source) is defined by the equations

$$\lambda \frac{\partial^2 T(x,t)}{\partial x^2} = \rho c \frac{\partial T(x,t)}{\partial t} \tag{1}$$

(heat conduction in the centre-Fourier's equation)

$$-\lambda \left.\frac{\partial T(x,t)}{\partial n}\right|_{x\in \text{border}} = \alpha \left(T(x,t) - T_{\text{environment}}\right)\Big|_{x\in \text{border}} \tag{2}$$

(heat exchange on the border-Newton's equation)

$$T(x,0) = T_{\text{initial}} \tag{3}$$

where the positive coefficients λ, ρ, c and α we receive as a constant.

Classical difference methods lead to the linear systems equations

$$\lambda \frac{T_{i-1l} - 2T_{il} + T_{i+1l}}{\Delta x^2} = \rho c \frac{T_{il} - T_{il-1}}{\Delta t} \tag{4}$$

where $1 \le i \le m$ and $l \ge 1$ (internal case)

$$\begin{cases} \lambda \dfrac{T_{1l} - T_{\text{env}}}{2\Delta x} = \alpha\left(T_{0l} - T_{\text{env}}\right) \\ \lambda \dfrac{T_{\text{env}} - 2T_{0l} + T_{1l}}{\Delta x^2} = \rho c \dfrac{T_{0l} - T_{0l-1}}{\Delta t} \end{cases} \tag{5}$$

where $l \geq 1$ and $T_{\text{env}} > T_{0l} > T_{1l}$ (border case)

$$T_{i0} = T_{\text{ini}} \tag{6}$$

where $0 \leq i \leq m$ (initial case)

2. Solution of the problem

From the linear system (5) we calculate only T_{0l} with the limitation

$$\frac{1}{2}\frac{\lambda}{\alpha} < \Delta x < \frac{\lambda}{\alpha} \tag{7}$$

In fact, the first equation from the system (5) gives

$$T_{1l} = T_{\text{env}} - \frac{2\alpha}{\lambda}\Delta x\left(T_{\text{env}} - T_{0l}\right) \tag{8}$$

so

$$T_{0l} - T_{1l} = \left(T_{0l} - T_{\text{env}}\right) + \frac{2\alpha}{\lambda}\Delta x\left(T_{\text{env}} - T_{0l}\right) = \left(T_{\text{env}} - T_{0l}\right)\left(\frac{2\alpha}{\lambda}\Delta x - 1\right) \tag{9}$$

and because $T_{\text{env}} - T_{0l} > 0$ and $T_{0l} - T_{1l} > 0$, that is

$$\frac{2\alpha}{\lambda}\Delta x > 1 \text{ or } \Delta x > \frac{\lambda}{2\alpha} \tag{10}$$

And now the linear system (5) gives

$$\begin{aligned} &\alpha T_{0l} - \frac{\lambda}{2\Delta x}T_{1l} = \left(\alpha - \frac{\lambda}{2\Delta x}\right)T_{\text{env}} \\ &\left(\frac{\rho c}{\Delta t} + \frac{2\lambda}{\Delta x^2}\right)T_{0l} - \frac{\lambda}{\Delta x^2}T_{1l} = \frac{\lambda}{\Delta x^2}T_{\text{env}} + \frac{\rho c}{\Delta t}T_{0l-1} \end{aligned} \tag{11}$$

with the determinant condition

$$\begin{vmatrix} \alpha & -\dfrac{\lambda}{2\Delta x} \\ \dfrac{\rho c}{\Delta t}+\dfrac{2\lambda}{\Delta x^2} & -\dfrac{\lambda}{\Delta x^2} \end{vmatrix} = \frac{\lambda}{2\Delta x}\begin{vmatrix} \alpha & -1 \\ \dfrac{\rho c}{\Delta t}+\dfrac{2\lambda}{\Delta x^2} & -\dfrac{2}{\Delta x} \end{vmatrix} > 0 \tag{12}$$

It only needs to point out that the determinant on the right side

$$\begin{vmatrix} \alpha & -1 \\ \dfrac{\rho c}{\Delta t}+\dfrac{2\lambda}{\Delta x^2} & -\dfrac{2}{\Delta x} \end{vmatrix} = -\frac{2\alpha}{\Delta x}+\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} \tag{13}$$

is positive, when

$$-\frac{2\alpha}{\Delta x}+\frac{2\lambda}{\Delta x^2} = \frac{2}{\Delta x}\left(\frac{\lambda}{\Delta x}-\alpha\right) > 0 \tag{14}$$

so

$$\frac{\lambda}{\Delta x}-\alpha > 0 \text{ or } \Delta x < \frac{\lambda}{\alpha} \tag{15}$$

The solution T_{0l} of linear system (5) has then the next intermediate form

$$T_{0l} = \frac{\dfrac{\rho c}{\Delta t}}{\dfrac{\rho c}{\Delta t}+\dfrac{2}{\Delta x}\left(\dfrac{\lambda}{\Delta x}-\alpha\right)} T_{0l-1} + \frac{\dfrac{2}{\Delta x}\left(\dfrac{\lambda}{\Delta x}-\alpha\right)}{\dfrac{\rho c}{\Delta t}+\dfrac{2}{\Delta x}\left(\dfrac{\lambda}{\Delta x}-\alpha\right)} T_{\text{env}} \tag{16}$$

For the internal linear system (4) and from symmetry condition $T_{m+1l} = T_{m-1l}$ we receive

$$\begin{gathered}
\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)T_{1l} - \frac{\lambda}{\Delta x^2}T_{2l} = \frac{\lambda}{\Delta x^2}T_{0l} + \frac{\rho c}{\Delta t}T_{1l-1} \\
-\frac{\lambda}{\Delta x^2}T_{1l} + \left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)T_{2l} - \frac{\lambda}{\Delta x^2}T_{3l} = \frac{\rho c}{\Delta t}T_{2l-1} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
-\frac{\lambda}{\Delta x^2}T_{m-2l} + \left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)T_{m-1l} - \frac{\lambda}{\Delta x^2}T_{ml} = \frac{\rho c}{\Delta t}T_{m-1l-1} \\
-\frac{2\lambda}{\Delta x^2}T_{m-1l} + \left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)T_{ml} = \frac{\rho c}{\Delta t}T_{ml-1}
\end{gathered} \tag{17}$$

with the positive determinant (see [1, 2])

$$\mathrm{DET}=\begin{vmatrix} \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & 0 & . & . & . & . & . \\ -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & 0 \\ . & . & . & . & 0 & -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} \\ . & . & . & . & 0 & 0 & -\frac{2\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} \end{vmatrix}_{m\times m} \tag{18}$$

$$\mathrm{DET}=D_m-\frac{\lambda}{\Delta x^4}D_{m-2} \tag{19}$$

where

$$D_j=\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^j-\binom{j-1}{1}\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^{j-2}\frac{\lambda^2}{\Delta x^4}+\binom{j-2}{2}\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^{j-4}\frac{\lambda^4}{\Delta x^8}-\ldots \tag{20}$$

Finally, we can write

$$\mathrm{DET}=\begin{vmatrix} \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & 0 & . & . & . & . & . \\ -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & 0 \\ . & . & . & . & 0 & -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} \\ . & . & . & . & 0 & 0 & -\frac{2\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} \end{vmatrix}_{m\times m} \tag{21}$$

$$\mathrm{DET}=\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^m-m\left(\begin{array}{l}\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^{m-2}\frac{\lambda^2}{\Delta x^4}-\frac{m-3}{2!}\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^{m-4}\frac{\lambda^4}{\Delta x^8}+\\ \frac{(m-5)(m-4)}{3!}\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^{m-6}\frac{\lambda^6}{\Delta x^{12}}-\\ \frac{(m-7)(m-6)(m-5)}{4!}\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)^{m-8}\frac{\lambda^8}{\Delta x^{16}}+\ldots\end{array}\right) \tag{22}$$

Next, the algebraic complements of the matrix

$$\begin{bmatrix}\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & 0 & . & . & . & . & .\\ -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ . & . & . & . & -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2} & 0\\ . & . & . & . & 0 & -\frac{\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2} & -\frac{\lambda}{\Delta x^2}\\ . & . & . & . & 0 & 0 & -\frac{2\lambda}{\Delta x^2} & \frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\end{bmatrix}_{m\times m} \tag{23}$$

there are the simple induction formes (see [2])

$$\begin{aligned}&A_{kk+p}=(-1)^{2k+p}\left(-\frac{\lambda}{\Delta x^2}\right)^p D_{k-1}\left(D_{m-k-p}-\frac{\lambda^2}{\Delta x^4}D_{m-k-p-2}\right)\quad \text{for } 0\le p\le m-k-1\\ &A_{km-1}=(-1)^{k+m-1}\left(\frac{\rho c}{\Delta t}+\frac{2\lambda}{\Delta x^2}\right)\left(-\frac{\lambda}{\Delta x^2}\right)^{m-k-1}D_{k-1}\\ &A_{mk}=2(-1)^{m+k}\left(-\frac{\lambda}{\Delta x^2}\right)^{m-k}D_{k-1}\quad \text{for}\quad 1\le k\le m-1\\ &A_{mm}=D_{m-1}\end{aligned} \tag{24}$$

according to the formula for D_j.

Also, the direct solutions of the linear system (17) are given

$$T_{il}=\frac{\left(\frac{\lambda}{\Delta x^2}T_{0l}+\frac{\rho c}{\Delta t}T_{1l-1}\right)A_{1i}+\ldots+\frac{\rho c}{\Delta t}T_{m-1l-1}A_{m-1i}+\frac{\rho c}{\Delta t}T_{ml-1}A_{mi}}{\mathrm{DET}} \tag{25}$$

for $1\le i\le m-2$

$$T_{m-1l}=\frac{\left(\frac{\lambda}{\Delta x^2}T_{0l}+\frac{\rho c}{\Delta t}T_{1l-1}\right)A_{1m-1}+\ldots+\frac{\rho c}{\Delta t}T_{m-1l-1}A_{m-1m-1}+\frac{\rho c}{\Delta t}T_{ml-1}A_{mm-1}}{\text{DET}} \tag{26}$$

$$T_{ml}=\frac{\left(\frac{\lambda}{\Delta x^2}T_{0l}+\frac{\rho c}{\Delta t}T_{1l-1}\right)A_{1m}+\ldots+\frac{\rho c}{\Delta t}T_{m-1l-1}A_{m-1m}+\frac{\rho c}{\Delta t}T_{ml-1}A_{mm}}{\text{DET}} \tag{27}$$

according to the formulas for A_{pq}.

References

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