ON PHILOSOPHICAL SENSE OF METAMATHICAL LIMITATIVE THEOREMS

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Logic was always considered as closely related to philosophy, even as an essential part of the latter. For example, Russell used to say that logic is the heart of philosophy. The connection of two fields can be illustrated by some views of Aristotle. According to the Stagirite the world consists of individual substances. Every substance has two elements, its matter and its form. Although the former particularizes substances, the latter brings generality. Due to the form, all human beings belong to the same kind (genus), because they share the property of being human. On the other hand, particular human exemplars differ with respect to their form. Now, one can investigate how this metaphysical or ontological idea is related to Aristotelian logic. As we know Aristotle constructed logic of categorical sentences of the form '… *S* is (are) … *P*', where the places marked by dots should be filled by 'All', 'No', some' or 'not'. As a result we have four forms 'All *S* are *P*', 'No *S* are *P*', 'Some *S* are *P*' and 'Some *S* are not *P*'. Take 'All *S* are *P*' illustrated as (a) 'All humans are mortal', for example. Presumably, the subject of (a), that is, the word 'humans' refers to particular human beings, but the predicate 'mortal' points out their essential property. Even if we disregard various controversial questions, like the nature of form or how it is bonded with matter in substances, we can still discuss whether Aristotle's ontological view determined his logic in the sense that taking categorical sentences as the basic logical forms was motivated by the doctrine that every substance consists of the matter and the form. However, the opposite direction is always open, because one can maintain that Aristotle passed from grammar to logic. The same problem appears in modern logic. In first-order logic, the form *Pa*, where the letter *P* functions as a predicate and the letter *a* is an individual name, that is, a proper name of an object. Now, one can say that whereas the name *a* refers to the matter, the component *P* expresses the form. And controversial issues appear once again. One can argue that so-called bare particulars, that is, pieces of pure matter (without properties) do not exist, although the opposite view is also present in philosophy. One will look for the route from logic to ontology, but others would like to insist that ontology is prior to logic. I do not intend to suggest any solution of the above philosophical controversies. This section only serves as an elementary introduction to the problem of how logic is related to philosophy or how we can look for the philosophical sense of logical constructions.

Today, metalogic or metamathematics plays a central role in logic, more precisely, in logic *sensu largo*, that is, covering the foundations of mathematics as well. Metamathematics (I will use this label) is a relatively new field. It was proposed by David Hilbert in the 1920s and was rapidly developed by him and his school, Thoralf Skolem, Kurt Gödel, Alfred Tarski, Alonzo Church and Alan Turing in the 1930s. Early metamathematical investigations culminated in famous limitative theorems. Typically, they include:

(The Löwenheim-Skolem Theorem; **LS**)

Every first-order theory has an infinitely countable model.

(The Gödel completeness theorem; **GCT**)

Every valid formula of a first-order theory ***T*** is provable in it (or equivalently: A first-order ***T*** theory is consistent if and only if it has a model); every first-order theory is semantically complete.

(The first Gödel incompleteness theorem; **1GT**)

If ***T*** is a consistent formal theory sufficient for expressing Peano arithmetic (**PA** thereafter) in it, ***T*** is incomplete, that is, one can construct formulas *G* and not-*G* which are not provable in ***T***; ***T*** is syntactically incomplete.

(The second Gödel incompleteness theorem; **2GT**)

If ***T*** is a consistent formal theory sufficient for expressing **PA** in it, its consistency cannot be proved in **T**.

(The Tarski undefinability theorem; **TT**)

If ***T*** is a consistent formal theory sufficient for expressing **PA** in it, the set of true sentences of ***T*** is not definable in it.

(The Church theorem of the undecidability of predicate calculus and **PA**; **CT**)

First-order predicate calculus and Peano arithmetic are undecidable, that is, there is no mechanical procedure (algorithm) for deciding which formulae of ***T*** are provable (valid).

Some comments are in order here. Firstly, the adjective 'limitative' suggest that these theorems exhibit some limitations of deductive formalized theories. Take **LS** as an example. We can formalize set theory as first-order. Consequently, all theorems about uncountable sets are provable in that theory. Let *t* be such a theorem. Since our theory has a countable model, *t* is true in it. This looks paradoxically (the Skolem paradox), because it seems that *t* requires an uncountable model for its truth. The paradox is resolved by observing that first-order theories are not sufficiently powerful in order to define some concepts inside them, for instance, relations between countable and uncountable sets. Thus, first-order formalisms are essentially limited as far as the matter concerns expressive power. Secondly, strictly speaking, all the listed theorems concern formalized theories, that is, expressed in formal languages. This property is essential only to the effect that rigorous proofs for these theorems would be impossible for non-formalized theories. On the other hand, we can give informal explanations. Take **1GT**, for example. Consider the sentence (b) 'I am unprovable in ***T***'. It asserts its own unprovability. Assume that (b) is true. If so,

(b) is unprovable in ***T***. Assume that (b) is false and our logic is sound, that is, does not prove falsehoods. Thus, if (b) is false, it cannot be provable. However, this reasoning does not constitute a proof in the exact mathematical sense. What Gödel did consisted in proving his theorems in a way admissible in mathematics via his famous technique of arithmetization. Secondly, all the listed theorems assume that our logic is classical (two-valued). The problem of how the situation looks like if we adopt a different logic, for example, intuitionistic or many-valued, is fairly complicated and must be skipped here. Let me only remark that changing the logic does not overcome the mentioned limitation limitations.

Thirdly, we assume that all theorems concern theories with countable languages (the number of symbols is denumerably infinite at most), finite formulas (every formula has a finite length) and finite rules of inference, that is, having a finite number of premises. Perhaps the last issue is more important. We can add the so-called ω-rule to ***T*** covering arithmetic. This rule allows us to infer the formula $\forall x A(x)$ from the infinite list of premises $A(a_1)$, $A(a_2)$, $A(a_3)$, …. . This addition is not trivial, because ***T*** + the ω-rule overcomes **1GT**, that is, becomes syntactically complete. Similarly, the ω-rule allows us to overcome **2GT**. We can look at **1GT** as asserting that the concept of natural number is fully axiomatized by first-order **PA**. This situation can be removed by an appeal to infinity, but we must add that this appeal requires an effective operating with infinite rules. However, this ability seems to be inaccessible for human minds. Finite rules of inference work in such a way that derivability is compact in the following sense: if A is derivable from a set ***X***, there is a finite subset ***Y*** of ***X*** such that A is derivable from ***Y***. This reasoning, provided that we cannot effectively grasp infinite sets of premises by our mental acts, means that we are still able to speak about infinities in terms of finite formulas. Needless to say, the last few sentences contains considerably rich philosophical content. An interesting fact is that adding the ω-rule to ***T*** does not overcome **TT**, because this supplement does not suffice for defining the concept of truth for arithmetic.

Fourthly, **CT** assumes the Church-Turing thesis (**CTT**). It can be stated as the assertion that a function is decidable (computable) in an intuitive sense if and only if it is recursive (I omit some subtleties related to the concept of recursivity). **CTT** is usually understood not as a mathematical theorem but rather as a proposal to equate the intuitive concept of decidability in a finite number of steps (that is, using an algorithm as a mechanical procedure or a Turing machine) with the exact mathematical concept of recursivity; this latter concept can be formalized in the arithmetic of natural numbers; in fact, Robinson's system **Q**, weaker than **PA**, is enough. Since the implication 'if a function is recursive, it is decidable' is trivial, the reverse dependence 'if a function is decidable, it is recursive' constitutes the hard core of **CTT**. Thus, we can say that the undecidability of first-order logic or arithmetic consists in the fact that related sets of their theorems or, using **GCT**, validities are not recursive. Finally, if we strengthen ***T*** by adding new axioms, for example, formerly unprovable sentences (e. g. the statement '***T*** is consistent'), we do not improve the situation, because a new unprovable formula can be immediately produced, for example, the assertion that a richer theory is consistent. Summing up, limitations suggested by metamathematical limitative theorems are actually essential and cannot be removed by typical mathematical manoeuvres. Clearly, any appeal to effectively graspable infinite rules of inference or infinitely long formulas exceeds the ordinary tools of mathematics.

Fifthly, all the above mentioned limitative theorems, in particular, **GCT**, **1GT**, **2GT** and **TT** shed light on how syntax is related to semantics. **GCT** might suggests that the syntactic description of formal systems, related to the concept of proof, and their semantic description, related to the concept of validity, are equivalent, because every provable formula is valid and reversely; this dependence concerns validity in all models in general (logical truth) as well as validity in all models of a given theory. The above mentioned intuitive demonstration of **1GT** uses the concept of truth. Gödel wanted to eliminate this appeal to truth and proceed by purely syntactic devices in his rigorous proof. He succeeded and his proof was constructive. When semantics, due to the works of Tarski became fully legitimate in metamathematics, **1GT** very often is formulated as **1GT'**: 'if ***T*** (I recall that ***T*** contains arithmetic) is consistent, there exist true but unprovable sentences expressible in its language'; this formulation provided **1GT** in a semantic setting. Thus, we can say that the concept of truth transcends the concept of proof in every concrete theory in which arithmetic is expressible. **TT** suggests more, namely that the concept of truth, which is the central semantic notion, transcends syntactic conceptual resources. This is the most important reason why adding infinite inference rules, which can be considered as syntactic devices, does not remove the undefinability of truth. Summing up, although arithmetical (that is, expressible in the language of arithmetic) syntax is fully constructive, the semantics of ***T***-theories is not, but enriching syntax by infinite rules, sometimes indispensable for semantic completeness, although it makes constructivity very problematic, does not suffice for semantics.

Sixthly, if we compare **GCT** and **CT**, we immediately observe a very deep difference between provability and decidability which does not reduce itself to the circumstance that some (un)provable) formulas exist, but (un)decidability concerns sets of formulas. Otherwise speaking, undecidable are theories, not single formulas. Perhaps the most important fact is that provability cannot be exhausted by decidability. In fact, a set ***X*** is recursive if and only if its complement is recursive as well. Now, it was proven that the non-recursivity of the set of non-validities of first-order logic and arithmetic is the actual cause of their undecidability. This means that theorems and non-theorems are not separable in general. There are exceptions, for example, propositional calculus or Presburger arithmetic (this is, **PA** but with addition as the sole binary operation), but the undecidability of theories is a fairly general phenomenon. Now, due to **GCT**, although theoremhood (what is provable) and truthhood (what is true) exactly correspond with respect to every first-order theory ***T***, provability and decidability essentially differ. One can eventually be inclined to say that the difference between provability and decidability is a derivative of the equivalence, even achieved by the ω-rule and similar infinitary devices, of the former and validity, but, on the other hand, decidability defined via recursivity is at odds with infinite rules of inference. Anyway, **CT**, although indirectly, exhibits the same difference between syntax and semantics which is suggested by **1GT** and **TT**. Incidentally, the gap between provability and undecidability shows why Leibniz's idea of an exact *lingua universalis* that would allows to solve every problem by *calculemus* was unrealistic.

I already marginally noted some philosophical aspects of limitative theorems. Now, I pass to a more systematic discussion, although the scope of this paper does not admit an exhaustive treatment (see Krajewski 2003 and Franzén 2007 for a more comprehensive presentation). The importance of metamathemathical results for the Hilbert program is widely recognized. Hilbert intended to create secure foundations for mathematics. More specifically, his program postulated a finitary proof of consistency of mathematics. He observed that proving that arithmetic is consistent would be entirely sufficient for the whole of mathematics. Unfortunately, **1GT** and **2GT**, eventually supplemented by **CT**, show that Hilbert's project is fairly unrealistic. If we take arithmetic as the upper bound of finitary methods, the fact (see **2GT**) that the assertion 'arithmetic is consistent' cannot be proved in arithmetic itself is fatal for the Hilbert program. Although partial realizations of Hilbert's ambitious project were effectively executed in the so-called reverse mathematics, his full proposal should, according to widespread opinion, be considered as a dream. In this way, limitative theorems essentially contributed to contemporary discussions in the foundations and philosophy of mathematics.

Another interesting lesson to be derived from metamathematics concerns the relation between formal and informal thinking. Doubtless, everything, that is, every mathematical theory can be formalized. This concerns not only typical theories, but also systems admitting infinitary formulas or languages with uncountable alphabets. However, any formalization is carried out in a language. Let ***L*** be a language to be formalized. This process proceeds in another language, let us say, ***ML***, called the metalanguage with respect to ***L***. This latter language cannot be fully formalized. The practice of metamathematics shows that ***ML*** is the language of ordinary mathematics, in particular, set theory. If we formalize ***ML***, which is always possible, we must do that in ***MML***, which is partly informal. Thus, if we consider the problem of the mutual relations between formal and informal elements in mathematics as a philosophical, perhaps epistemological question, metamathematics suggests that the latter are simply indispensable. On the other hand, rigorous proofs of limitative (as well as other metamathematical) theorems would arew probably impossible without the formalization of deductive systems. As a by-product of this reasoning we have the conclusion that, if the ***ML*** used for formalization of logic itself employs ordinary mathematical devices, logical theory is not prior to mathematics. Mathematics and logic go together in informal discourse.

Both philosophical conclusions were presented in the last section as almost indubitable. Yet 'almost' must be taken very seriously. In fact, although most mathematicians and philosophers agree that **1GT** and **2GT** effectively demolished the Hilbert program, there are people who think otherwise (see Detlefsen 1986 for a more comprehensive discussion). The main argument points out that the concept of finitary methods is liable to various interpretations. Now, if we adopt a liberal interpretation, we can accept Gentzen's proof of the consistency of arithmetic as finitary. This proof uses so-called restricted transfinite induction, that is, induction to the ordinal ε_0, that is, the limit of the sequence ω, ω^ω, …. . The proponents of this view agree that restricted transfinite induction is not formalizable in **PA**, but they consider it as remaining in the finite frameworks. As far as the matter concerns the relation between the formal and the informal, one can argue that informal elements in mathematics can be always eliminated in favor of full formalization, because they are only temporary and heuristic. This route of thinking will stress the fact that although there is no possibility to formalize everything in one stage, every level is subjected to a fully formal treatment. The same concerns the question of whether logic is prior to mathematics or not. Thus, the situation is fairly paradoxical. On the one hand, we have very well-established and rigorously proven metamathematical results, but, on the other hand, their philosophical interpretation is not univocal. Even more, these hard results generate mutually inconsistent philosophical opinions. But how is this possible?

In order to try to answer to the last question let me consider an example from physics. It is frequently said that the uncertainty principle (the Heisenberg rule) in quantum physics implies indeterminism, contrary to deterministic classical mechanics. This principle can be formally expressed by the nonequality (*) $\Delta p_1 \Delta p_2 \geq h$, where Δp_1, Δp_2 and h refer to the uncertainty of position, the uncertainty of momentum (both related to measurement processes) and Planck's constant; respectively. This means that a simultaneous fully measurement of the position and momentum of quantum objects is impossible. Hence, we are not able to a fully precise description of the initial state of an object. Consequently, this uncertainty pertains to predictions of future states of quantum objects and their complexes. The standard interpretation of quantum theory says that these predictions are indispensably probabilistic. What about determinism and indeterminism in this context? First of all, we must observe that the term 'determinism' and 'indeterminism' do nor not occur in (*). Hence, provided that implication is understood exactly, that is, as a mark of deduction, (*) cannot imply indeterminism, because the conclusion of deductive reasoning cannot contain terms not occurring in the premises or not defined by conceptual resources generated by the assumptions of a given inference. Thus, we should define indeterminism and determinism. Heisenberg did this by saying that a theory is deterministic if its equation allow us to calculate exactly the future state of an object on the basis of a description of its initial state, otherwise it is indeterministic. Consequently, quantum mechanics implies indeterminism, but classical mechanics entails determinism, because the latter is deterministic, but the former indeterministic. This is a brief summary of what quantum mechanics brought into philosophy.

Yet much more can and should be said about the reasoning in the last section, that is, the derivation of ontological indeterminism from the uncertainty principle. First of all, physicists *qua* physicists have no need to worry about quantum theory (or any other physical theory) is deterministic or not. Otherwise speaking, the philosophical interpretation of (*) nothing adds to the physical content of quantum mechanics. This theory is a part of physics and serves physical tasks. When Heisenberg argued that his principle entails indeterminism from (*), he acted rather as a philosopher than a physicist, or, more carefully, as someone working on the borderline between physics and philosophy. Secondly, the philosophical nature of the inference from (*) to determinism explains why other physicists (Albert Einstein, Louis de Broglie or David Bohm) looked for deterministic interpretations of quantum physics and considered the standard theory (the Copenhagen interpretation) as only temporary (see Auletta 2000; this massive book shows how complex the issue in question is). In fact, the situation is very similar to that in metamathematics: we have hard, commonly accepted physical theories and their various, sometimes mutually inconsistent philosophical interpretations. In the case of the reasoning in question, (*) is the hard core and the definition of (in)determinism its philosophical aspect. Now a natural question arises: how are these two components related?

In my view, so-called philosophical conclusions derived from mathematical or physical theorems are not simple logical (deductive) consequences of related theorems of mathematics or physics, but the results of very complicated patterns of reasoning. Two components of such actions should be sharply distinguished. One consists in taking a scientific theorem as the starting point. Then, we must do some interpretative work, philosophical in principle. It is not easy to describe this stage in a general manner and concrete cases should be treated separately, at least at the beginning. Take the reasoning from the principle (*) to indeterminism once again. Heisenberg had to define (in)determinism in some way. The important point is that he could not disregard the philosophical tradition. On the other hand, his proposal regarding how to define (in)determinism was explicitly related to the physical content of quantum mechanics. However, this relation is not formal or logical. I see no other way to characterize the step from the physical content of a theory ***T*** to its philosophical elaboration than to say that it essentially requires a certain amount of hermeneutic (or understanding if this category is preferable). Otherwise speaking, we must be able to embed some theorems of ***T*** into philosophical concepts, theories and categories. This procedure equips ***T*** (or at least some of its elements) with philosophical sense and importance. For example, in the case of (*) we have at our disposal the long philosophical tradition related to determinism and indeterminism how physical theories were philosophically interpreted, for example, how it was done with respect to classical mechanics and statistical physics. In particular, the effective reduction of thermodynamics (the paradigm of statistical physics) to classical mechanics could suggest that every probabilistic element in science is to be eliminated. The consequences of hard mathematical or physical (I do not limit the scope of the hard core) theorems via hermeneutic interpretation can be called interpretative. Formally speaking, they can always be made deductive, but the main job consists in embedding premises into a hermeneutic context. In fact, the results of such embeddings are rather philosophical paraphrases of scientific assertions than their translations into scientific language. This part of the philosophical work depends of many factors, for example, traditions, expectations, etc. Although no rigid rules are available for performing hermeneutical embeddings of definite scientific results, the idea of interpretative consequences explains why the same hard cores lead to inconsistent philosophical conclusions.

Let me return to metamathematics now. I will illustrate the foregoing analysis of interpretative consequence by the problem of the epistemological status of the theorems of logic and mathematics, that is, assertions belonging to the formal sciences. One view (logical empiricism) maintains that formal truths are analytic, an other (Kant) that they are synthetic a priori, still another (J.S. Mill) that they are synthetic a posteriori, that is, empirical. Clearly, this issue concerns the cognitive value of formal science and might be regarded as an exercise in general philosophy, not only philosophy of logic and mathematics. Consider the following argumentation against the analytic theory of mathematical truths (see Woleński 1993, Woleński 2003a, Woleński 2004 for further details). Define analytic truths as derivable by purely logical devices only; this account was proposed by Gottlob Frege. Take G as a Gödelian sentence, that is, unprovable in arithmetic; its existence is guaranted by **1GT**. Consequently, its negation not-G is also a Gödelian sentence. Since our logic is classical, one of the pair $\{G$, not-$G\}$ is true; eventually, one can directly use **1GT'**. Thus, we have a true but unprovable sentence. It is not analytic in virtue of the adopted definition of analyticity as derivability by purely logical procedures. Hence, there are arithmetical truths which are not analytic and the analytic account of mathematics fails. However, since **1GT** says nothing about analyticity, we must perform hermeneutic work in order to link this theorem with the problem of the cognitive status of mathematics as explored via the concept of analytic sentences. On the other hand, a defender of the analytic account of mathematics will either modify the definition of analyticity or adopt so-called ifthenism, that is, the view, suggested by Russell, that, mathematics consists of conditional assertions of the form $\boldsymbol{AX} \Rightarrow A$, where AX represents the conjunction of axioms of a given mathematical theorems. We have, via the deduction theorem, (**) if $\boldsymbol{AX} \rightarrow A$, then $\rightarrow (\boldsymbol{AX} \Rightarrow A)$, where the symbol $\rightarrow$ refers to provability. Thus, (**) means 'if A is provable from $\boldsymbol{AX}$, then $\boldsymbol{AX} \Rightarrow A$ is provable in logic. Since we can always add G as well as not-G (both belong to the language of mathematics) as a new axiom and form a richer theory, the formulas $\boldsymbol{AX'} \Rightarrow G$ and $\boldsymbol{AX''} \Rightarrow$ not-G are trivially provable in pure logic, that is, we have $\rightarrow (\boldsymbol{AX'} \Rightarrow G)$ and $\rightarrow (\boldsymbol{AX''} \Rightarrow G)$. I do not suggest which account is correct. My only taim was to show that the same limitative theorem can be use at one time for a criticism of a philosophical position and another time for its defense.

My last example concerns a typical great philosophical problem. According to Berkeley, bodies are complexes of subjective qualities, sense-data using modern terminology. Thus, bodies exist in the consciousness of particular subjects. Yet Berkeley argued that he had ordinary bodies in his mind. This position, called immanent or subjective idealism, is famously summarized by Berkeley's dictum (#) *esse = percipi.* Contrary to Berkeley's view that (#) is fully coherent with the ordinary point of view, subjective idealism is just considered as being at odds with commonsensical intuitions. One of the criticisms of Berkeley's account of bodies runs as follows (Ajdukiewicz 1948). Since Berkeley defined bodies by perceptual subjective contents, he employed only relations between sense-data. In particular, sense-data do not refer to anything external. Thus, Berkeley employed a language very similar to the language of syntax, which elaborates relations between expressions. Since, according to Ajdukiewicz, our ordinary talk about bodies refers to them as external entities, it is semantic in its nature. Berkeley's reasoning can be presented as an attempt similar to defining the semantic aspect of language by purely syntactic resources. However, limitative theorems show that semantics is not definable by pure syntax. By analogy, we can conclude that Berkeley's account fails. We have another version of the same argument (proposed by Roman Suszko in his unpublished lecture delivered in Krakow in 1964). Take (#) as a definitional equality with *esse* as the definiendum and *percipi* as the definiens. Interpret the definiens as syntactic and the definiendum as semantic. The further argument is the same as in Ajdukiewicz 1948. The essence of this defense of epistemological realism against subjective idealism is this. At first, we establish an analogy between epistemology and syntax/semantics and observe that we need a decision concerning which kind of metalanguage is to be used. Next, we observe that a philosopher should clearly explain whether he or she employs the syntactic or semantic metalanguage. If the former is chosen, idealism is unavoidable. On the other hand, if ordinary intuitions are to be obeyed, the semantic metalanguage must be used, but this leads to realism.

Evert W. Beth (see Beth 1968, s. 620) reported Ajdukiewicz's argument against idealism and qualified it as conclusive. Although I share Ajdukiewicz's method and his defense of epistemological realism (see Woleński 2007), I think that Beth's qualification is too strong. In fact, only those philosophers who agree with Ajdukiewicz about paraphrasing epistemological problems into the language of metamathematics and using limitative theorems will approve his argument as conclusive. Clearly, a massive interpretative work must be done in order to prepare the realism/idealism controversy to be settled by semantics. In this case, the amount of hermeneutic is much greater than in the case of discussing the issue of whether the truths of formal sciences are analytic or synthetic. Thus, we are not able to foresee in advance how much hermeneutic work is required in order to base philosophical arguments on metamathematics; this remark concerns philosophical uses in other sciences as well.

Although we can expect that issues belonging to the philosophy of mathematics require less hermeneutic than problems from classical ontology and epistemology, no general measure is available. Hence, if we ask what is the philosophical sense of metamathematical results, no straightforward answer is possible. Surely, limitative theorems (and others as well) have their literal or legal sense in their proper domains of application. This sense creates possibilities concerning how philosophers can use metamathematics in their arguments. However, the most essential part of philosophical work with limitative theorems consists in hermeneutic interpretations. This explains why rigid rules in this domain do not exist. Although metamathematics does not provide a new philosophical stone, on the other hand, it does suggest a new and interesting language for old problems.

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