STABILITY AND EQUILIBRIA IN THE MATCHING MODELS

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Abstract. We define generalized market equilibria for the matching model of Gale and Shapley [1] and study relationships between the notion of equilibrium and the notion of stable matching of Gale and Shapley. Namely, we show that there is a one-to-one correspondence between full-quota matchings and the so-called simple equilibria and a one-to-one correspondence between stable matchings and the so-called stable equilibria.

Introduction

In the paper [1] Gale and Shapley modelled the college admission process as a process of matching students with colleges with the crucial role of preferences of both sides of the "market". Gale-Shapley model became very popular in economics - for example it was used for modelling different two-sided matching markets, eg labour market, see [2], or auction market, see [3].

Gale-Shapley theory is based on the notion of stable matching. Such kind of matching assigns students to colleges such that no matched pair (student, college) have an incentive to change their situation (see section 1). Hence, stable matching may be treated as some kind of equilibrium in the matching market.

There are different notions of equilibrium in economic theory. Typical ones are the notions of partial or general (Walrasian) market equilibrium (see e.g. [4], chapters 4, 5, and [5]), when interactions between buyers (consumers) and sellers (producers) of some good (goods) are studied and demand of the buyers is compared with supply of the sellers. In such models market equilibrium is determined by a price system for which demand and supply are equal.

It appears that in the matching (college admissions) model we can very easily define demand and supply, but there are no explicitly stated prices there. Yet we can define equilibrium in such a model by introducing some generalization of the classical market approach. Our generalized approach is based on the notion of conditions (requirements) stated by the sellers, which guarantee any buyer to obtain a given amount of good offered by a seller. Observe that stating a fixed price of a good by a seller can be treated as a special case of such a condition, which can be explicitly formulated as follows: "You can obtain a given amount of good offered by me under the condition that you will pay me a given amount of money".

Assume that in the matching market each college states some condition for a student, which guarantees admitting him/her to this college (for example minimal number of exam scores needed to be admitted). It appears that in this situation we can easily define students' demand and equilibrium in our matching model (see section 2).

The aim of our paper is to give a precise definition of generalized equilibrium for the matching market (in the sense explained above) and to study relationships between such defined equilibria and stability. Namely, we prove (in section 3) that there is a one-to-one correspondence between full-quota matchings and the so-called "simple" equilibria and (in section 4) that there is one-to-one correspondence between full-quota stable matchings and the so-called "stable" equilibria. The final conclusion is that stable matchings can be, in fact, interpreted as some kind of generalized market equilibria for the matching market.

1. Matching model

Let S be a finite, nonempty set of students (buyers), and C – a finite, nonempty set of colleges (sellers). For any student $s \in S$ we define a nonempty set $A(s) \subset C$ interpreted as a set of colleges acceptable for s, and for any $c \in C$ we define a nonempty set $A(c) \subset S$ being a set of students acceptable for c. Each student $s \in S$ has fixed preferences in the set $A(s)$ defined by some strict linear order $>_s$. Similarly, each college $c \in C$ has preferences in $A(c)$ defined by a strict linear order $>_c$.

We also assume that each college $c \in C$ has a quota number $q_c \geq 1$ (q_c is the maximum number of students which can be admitted to c) and that each student may be admitted to no more than one college.

Throughout the rest of the paper we assume that the sets S, C, $A(s)$, $A(c)$, the orders $>_s$, $>_c$ and the numbers q_c are fixed.

We will use the following notation: the symbol sc will denote the unordered pair $\{s, c\}$, and the symbol $|X|$ - cardinality of a (finite) set X.

Let

$$E(S,C) = \{sc: s \in S, c \in C\}$$

be the set of unordered pairs consisting of elements of the sets S and C (the set of edges in the complete bipartite graph connecting S and C). For any $\mu \subset E(S,C)$ we denote by

$$\mu(s) = \{c \in C : sc \in \mu\}$$

$$\mu(c) = \{s \in S:\ sc \in \mu\}$$

the sets of neighbouring vertices for s and c, respectively (if μ is a matching as in the Def. 1 below, then $\mu(s)$ is the set of colleges matched to s, and $\mu(c)$ is the set of students matched to c).

The symbol

$$A(S,C) = \{sc \in E(S,C): s \in A(c),\ c \in A(s)\} \tag{1}$$

will denote the set of (two-sided) acceptable pairs in $E(S,C)$ (it is possible that $A(S,C) = \varnothing$).

Definition 1. *Matching* is a set $\mu \subset A(S,C)$ such that

$$\text{(i)} \quad |\mu(s)| \le 1, \qquad \forall\, s \in S$$

$$\text{(ii)} \quad |\mu(c)| \le q_c, \quad \forall\, c \in C$$

Inequality (i) means that each student is matched to at most one college, and (ii) - that to each college c no more than q_c students are admitted.

Definition 2. A matching μ is a *full-quota matching* if

$$|\mu(c)| = q_c, \quad \forall\, c \in C$$

Now we are ready to define stable matchings (using the notion of blocking pairs).

Definition 3. A pair $sc \in A(S,C)$ is a *blocking pair for a matching* μ if the following holds:

$$\text{(i)} \quad c >_s \mu(s)$$

$$\text{(ii)} \quad |\mu(c)| < q_c \quad \text{or} \quad \exists\, r \in \mu(c):\ s >_c r$$

In (i) we identify $\mu(s)$ with the only college to which s is admitted under μ (if $\mu(s) \neq \varnothing$). We assume also that $c >_s \varnothing$ always holds (s prefers any acceptable college rather than being non-matched).

If a pair sc blocks a matching μ, then the student s prefers the college c to the college to which he/she is admitted and also the college c prefers s to some student r admitted to c (or c has some unfilled positions and s can be enrolled at this position). Hence both parties have an incentive to change their situation and disrupt the matching μ. Matchings for which such kind of disruption is impossible are called stable.

Definition 4. A matching μ is *stable* if there are no blocking pairs for μ.

2. The concept of equilibrium

Assume that each college $c \in C$ states some conditions (requirements) $R(c)$ such that satisfying these conditions by a student $s \in S$ guarantees him/her to be admitted to the college c. We will identify $R(c)$ with the set of students which satisfy the conditions $R(c)$. Hence, for any college $c \in C$, a set $R(c)$ is defined such that

$$R(c) \subset A(c) \subset S$$

We do not exclude here the case $R(c) = \varnothing$ (no student satisfies the conditions of the college c).

Let $Sub(S)$ be the family of all subsets of the set S.

Definition 5. A *system of conditions* is a mapping R: $C \to Sub(S)$ such that $R(c) \subset A(c)$ for each $c \in C$.

Definition 6. Let R be a system of conditions and let $s \in S$ be a student. A *set of feasible colleges for a student $s \in S$ under the conditions R* is the set

$$F_R(s) = \{c \in A(s) : s \in R(c)\}$$

$F_R(s)$ is the set of colleges such that the student s satisfies the conditions of each college from this set (it is possible that $F_R(s) = \varnothing$).

Definition 7. Assume that $F_R(s) \neq \varnothing$ for some student s. *The best college for s under the conditions R* is the college

$$F_R^*(s) = \max\,\{c\text{: } c \in F_R(s)\}$$

Here the maximum is taken with respect to the preference ordering $>_s$ of the student s. It is obvious that $F_R^*(s)$ is well defined unique college, because $>_s$ is a linear order in C and hence in $F_R(s)$.

For each college $c \in C$ and each system of conditions R we can also define the set

$$\varphi_R(c) = \{s \in A(c)\text{: } F_R^*(s) = c\}$$

which is the set of students which are acceptable for c and for which c is the best college under the conditions R. Observe that the sets $\varphi_R(c)$ are disjoint (for fixed R) and that $\varphi_R(c) \subset A(c)$, $\forall c \in C$.

We note here that the set of feasible colleges and the best college for a student s are notions analogous to the notions of the set of feasible bundles (consumption set, budget set) and the optimal bundle for a consumer in consumer theory. Hence the set $\varphi_R(c)$ can be treated as the set of students which constitute the demand for the positions offered by the college c. Thus the mapping $c \to |\varphi_R(c)|$ can be interpreted as a demand function for a system of conditions R in our "college market". On the other hand, the numbers q_c (quotas for colleges) determine the supply function: $c \to q_c$.

Comparing demand and supply we reach the notion of equilibrium in the matching market.

Definition 8. A system of conditions R is an *equilibrium* if

$$|\varphi_R(c)| = q_c, \quad \forall c \in C$$

Observe that if c is the best college for a student s under the conditions R, then, by the definition of $F_R(s)$ and $F_R^*(s)$, the student s satisfies the conditions $R(c)$. Hence

$$\varphi_R(c) \subset R(c), \quad \forall c \in C \tag{2}$$

If, for each college and for some equilibrium R, the above inclusion becomes an equality, then we obtain the so-called simple equilibrium.

Definition 9. An equilibrium R is a *simple equilibrium* if

$$\varphi_R(c) = R(c), \quad \forall c \in C$$

Since the sets $\varphi_R(c)$ are disjoint, so for a simple equilibrium the sets $R(c)$ are also disjoint. This means that each student satisfies the conditions of no more than one college. Hence, if a student s satisfies the conditions of a given college, then this college is automatically the best college for s under R.

In the next section we will need the following characterization of simple equilibria.

Lemma 1. *Let R be a mapping $C \to Sub(S)$. Then R is a simple equilibrium if and only if the following holds:*

(i) *The sets $R(c)$ $(c \in C)$ are disjoint*

(ii) $|R(c)| = q_c, \quad \forall c \in C$

(iii) $\forall c \in C\ \ \forall s \in S \quad s \in R(c) \Rightarrow sc \in A(S,C)$

Proof. If R is a simple equilibrium, then, by Def. 8 and 9, the sets $R(c)$ are disjoint and $|R(c)| = q_c$. To prove (iii) observe that $s \in R(c)$ implies $s \in A(c)$ (by Def. 5) and that it also implies $s \in \varphi_R(c)$ (by Def. 9). Hence $F_R^*(s) = c$, and since $F_R^*(s) \in F_R(s) \subset A(s)$, we obtain that $c \in A(s)$. Formulas $s \in A(c)$ and $c \in A(s)$ imply $sc \in A(S,C)$ (by (1)).

Assume now that a mapping R satisfies (i), (ii) and (iii). By (iii) we obtain $R(c) \subset A(c)$, $\forall c \in C$, hence R is a system of conditions. Since $R(c)$ are disjoint, any

s belongs to no more than one of the sets $R(c)$. Hence, if $s \in R(c)$, then $F_R(s) = = \{c\}$ (because $c \in A(s)$ by (iii)) and so $F_R^*(s) = c$. Therefore $s \in \varphi_R(c)$ (because $s \in A(c)$ by (iii)) and we obtain the inclusion $R(c) \subset \varphi_R(c)$. Thus, by (2), $\varphi_R(c) = = R(c)$, and, by (ii), $|\varphi_R(c)| = q_c$. Hence R is a simple equilibrium.

3. Full-quota matchings as market equilibria

Now we will prove that there is close relationship between full-quota matchings and simple equilibria. Namely, by Theorem 1 below we can identify any full-quota matching with some simple equilibrium in the matching market.

Theorem 1. *There is one-to-one correspondence between the set of full-quota matchings and the set of simple equilibria*

Proof. Let μ be any full-quota matching. We define a mapping $g(\mu)$: $C \to Sub(S)$ as follows:

$$g(\mu)(c) = \mu(c) \tag{3}$$

We will prove that $g(\mu)$ is a simple equilibrium. By Def. 1 and 2, we have

(i) $|\mu(s)| \leq 1, \quad \forall\, s \in S$

(ii) $|\mu(c)| = q_c \;\forall\, c \in C$

(iii) $\mu \subset A(S,C)$

By (i), all the sets $\mu(c)$, $c \in C$, are disjoint, and by (iii), $s \in \mu(c)$ implies $sc \in A(S,C)$. Hence, by Lemma 1, $g(\mu)$ is a simple equilibrium.

Let now R be any simple equilibrium. Define the set

$$h(R) = \{sc \in E(S,C):\; s \in R(c)\} \tag{4}$$

We will prove that $\mu = h(R)$ is a full-quota matching. By Lemma 1, the sets $R(c)$ are disjoint, hence $|\mu(s)| \leq 1$ ($s \in S$). We have also $|\mu(c)| = q_c$, because $|R(c)| = = q_c$ for all $c \in C$. Moreover, condition (iii) from Lemma 1 implies that $\mu \subset A(S,C)$. Hence, by Def. 1 and 2, μ is a full-quota matching.

Thus far, we have defined two functions: g - acting from the set of all full-quota matchings to the set of all simple equilibria, and h - acting from the set of all simple equilibria to the set of all full-quota matchings. To end the proof it suffices to show that the compositions $h \circ g$ and $g \circ h$ are identities. It is obvious if we use the following equalities:

$$(h \circ g)(\mu) = h(g(\mu)) = \{sc \in E(S,C) : s \in \mu(c)\} = \mu$$

$$(g \circ h)(R) = g\{\, sc \in E(S,C): s \in R(c)\} = R$$

4. Stable matchings and stable equilibria

In this section we define the so-called stable equilibria, which are special kinds of simple equilibria and show that they are in one-to-one correspondence with full--quota stable matchings. First we define the so-called improving extension of an equilibrium.

Definition 10. Let R be an equilibrium. A mapping T: $C \to Sub(S)$ is called an *improving extension* of R if

(i) $R(c) \subset T(c), \quad \forall c \in C$

(ii) $(\forall c \in C)\ (\forall s \in T(c) \setminus R(c))\ (\exists r \in R(c)) : \ s >_{c} r$

Defining an improving extension of R means that each college weakens the conditions R, but in such a way, that each student satisfying the new conditions T (and not satisfying the old ones) is better than at least one student satisfying the old conditions R.

Definition 11. Let R be a simple equilibrium. We call R a *stable equilibrium* if any improving extension of R is also an equilibrium.

Theorem 2. There is a one-to-one correspondence between the set of full-quota stable matchings and the set of stable equilibria.

Proof. We will use the functions g and h defined in the proof of Theorem 1. To prove Theorem 2 it suffices to show that:

(i) For any full-quota stable matching μ, a mapping $g(\mu)$, defined by (3), is a stable equilibrium.

(ii) For any stable equilibrium R, a matching $h(R)$, defined by (4), is a full-quota stable matching.

To prove (i) assume that μ is a full-quota matching, but $g(\mu)$ is not stable (by Theorem 1, $g(\mu)$ is a simple equilibrium). We will prove that μ is not stable. Let $R = g(\mu)$. By our assumption, R is not stable, hence there exists an improving extension of R, say T, which is not an equilibrium. It follows that for some college $c \in C$ we have $|\varphi_T(c)| \neq q_c$ and $|\varphi_R(c)| = q_c$ (because R is an equilibrium). This means that the sets $\varphi_T(c)$ and $\varphi_R(c)$ are distinct. Hence at least one of the following two cases is possible:

(A) There exists a student $s \in \varphi_T(c)/\varphi_R(c)$ (college c is the best college for s under the conditions T, but it is not the best college for s under the conditions R).

(B) There exists a student $s \in \varphi_R(c)/\varphi_T(c)$ (college c is the best college for s under the conditions R, but it is not the best college for s under the conditions T).

We will prove that in both cases (A) and (B) we can find a blocking pair for μ, thus showing that μ is not stable.

Consider first the case (A). We will prove that $c >_s \mu(s)$. If $\mu(s) = \varnothing$, the inequality is obvious, so assume that $\mu(s) \neq \varnothing$. Since $R = g(\mu)$ is a simple equilibrium, it follows that $\mu(s)$ is the only college in $A(s)$ for which the conditions R are satisfied by s. So it is the best college for s under the conditions R. By (A), c is not the best college for s under R, hence $c \neq \mu(s)$. We have $R \subset T$ (T is an improving extension of R) and so s satisfies the conditions T for the college $\mu(s)$. Since c is the best college for s under T, we have $c >_s \mu(s)$.

By (ii), Def. 10, we also obtain that there is a student $r \in R(c)$ such that $s >_c r$ (because $s \in T(c)/R(c)$). So finally we have $c >_s \mu(s)$ and $s >_c r$ for some $r \in R(c) = \mu(c)$ and thus, by Def. 3, the pair sc is a blocking pair for μ.

Consider now the case (B). Let b be the best college for s under the conditions T (such college exists because the best college for s under R exists and $R \subset T$). Since $R \subset T$ and c is not the best college for s under T, we have $b >_s c = \mu(s)$. By (ii), Def. 10, it follows also that there exists $r \in R(b) = \mu(b)$ such that $s >_b r$ (because $s \in T(b)/R(b)$). Thus sb is a blocking pair for μ.

Now we will prove (ii) (by contradiction). Let R be a simple equilibrium and $\mu = h(R)$ a full-quota matching which is not stable. We want to prove that R is not stable.

The matching μ has the form

$$\mu = \{sc \in E(S,C):\ s \in R(c)\}$$

From nonstability of μ it follows that some pair, say $sb \in A(S,C)$, is a blocking pair for μ. Hence s prefers b to $\mu(s)$ and b prefers s to some $r \in \mu(b) = R(b)$. Thus, if we define a mapping $T\colon C \to Sub(S)$ by:

$$T(c) = R(c), \quad \text{if} \quad c \neq b$$

$$T(b) = R(b) \cup \{s\}$$

then T is an improving extension of R. We want to prove that T is not an equilibrium. Observe that for the student s the college b is the best college under T and for the other students the same colleges are the best for both the conditions T and R. Hence $|\varphi_T(b)| = q_b + 1 \neq q_b$ and so T is not an equilibrium.

Thus we have proved that for an equilibrium R there exists an improving extension T which is not an equilibrium. Hence R is not stable and the proof of Theorem 2 is finished.

Conclusions

We have proved that there is close relationship between the notion of stable matching and the notion of "stable" equilibrium in the matching market. Here equilibrium is meant as some kind of classical market equilibrium, but without explicitly stated prices. The role of prices is played by some generally defined "conditions", which can be quite naturally stated, for example, in the college market. Hence, the well-known and very popular notion of stable matching can be interpreted as some kind of generalized market equilibrium in the matching market.

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