ANALYSIS AND OPTIMIZATION OF MARKOV HM-NETWORKS WITH STOCHASTIC INCOMES FROM TRANSITIONS BETWEEN THEIR STATES

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Abstract. Expressions for expected incomes and variations of incomes in systems of Markov HM-networks, when service rates of messages are depending or not depending of network's states are obtained. The case when incomes of transitions between network's states are random variables with the set moments of first two orders is considered. Solution of some optimization problems for HM-networks are resulted.

Introduction

It is known that functioning of any Markov queueing network (QN) is described with a help of Markovian chain with continuous time. In Howard's work [1] the conception of Markovian chains with incomes that were constants was introduced and there were proposed to use method of Laplace transformation and method of z-transformation for analysis of such chains with small number of states. This concept has laid down in a basis of definition of Markov QN with incomes that were examined in works [2-4] at first. Later open and closed networks with central queueing system (QS) were investigated in cases when a) incomes from transitions between network's states depend on states and time [5-7] or b) incomes are random variables (RV) with the set moments of the first and the second orders [8, 9]. Recently QN with incomes refer to as HM (Howard-Matalytski)-networks. In the present article expressions in case b) were obtained for expected incomes and income's variations in systems of open exponential HM-networks and Jackson HM-networks of arbitrary architecture.

Let us examine open exponential QN with one type messages that consist of n QS $S_1, S_2, ..., S_n$. Vector $k(t) = (k,t) = \left(k_1, k_2, ..., k_n, t\right)$ is state of network where k_i is number of messages in system S_i at the moment t, $t \in [t_0, +\infty)$, $i = \overline{1,n}$. For unification of designation let us introduce system S_0 (outside medium) from which the Poisson flow of messages with arrival rate λ comes into network. At first we will examine the case when service rate of messages μ_i in system S_i doesn't depend on number of messages in it, $i=\overline{1,n}$. Let p_{0j} is probability of message entry from system S_0 to system S_j, $\sum_{j=1}^{n} p_{0j}=1$; p_{ij} is probability of message transition after its service in system S_i to system S_j, $\sum_{j=0}^{n} p_{ij}=1$, $i=\overline{1,n}$. Message during its transition from system S_i to system S_j brings to system S_j some random income and income of system S_i descend on this value correspondently, $i,j=\overline{0,n}$. It is needed to find expected (average) incomes of network's systems during time t considering that network's state in the starting time t_0 is known. At first we will suppose that network is functioning in the term of high capacity, i.e. $k_i(t)>0$, $\forall t\in[t_0,+\infty)$, $i=\overline{1,n}$.

1. Expected incomes of network systems

Let us examine dymamics of income's change of some network's system S_i. Let us denote its income at the moment t as $V_i(t)$. The income of this system at some moment t_0 is equal to v_{i0}. We will be interested in system's income $V_i(t_0+t)$ at the moment t_0+t. To a finding of the income we shall apply a following technique. Let us divide interval $[t_0,t_0+t]$ into m equal portions with length $\Delta t=\frac{t}{m}$, and consider that m is big. For finding of system's income Let us write probability of all events that can occur at the l-th time interval, $l=\overline{1,m}$. The next situations are possible: a) message from outside will arrive to system S_i with probability $\lambda p_{0i}\Delta t+o(\Delta t)$ and will bring income of size r_{0i} to it, where r_{0i} – RV with distribution function (d.f.) $F_{0i}(x)$; b) message from system S_i will transmit with probability $\mu_i p_{i0}\Delta t+o(\Delta t)$ into outside medium and system's income will decrease on value R_{i0}, where R_{i0} – RV with d.f. $F_{i0}(x)$; c) message from system S_j will transmit with probability $\mu_j p_{ji}\Delta t+o(\Delta t)$ into system S_i and income of system S_i will increase on value r_{ji} and income of system S_j will decrease on this value, $j=\overline{1,n}, j\neq i$, where r_{ji} – RV with d.f. $F_{1ji}(x)$; d) message from system S_i will transmit with probability $\mu_i p_{ij}\Delta t+o(\Delta t)$ into system S_j and income of system S_j

will increase on value R_{ij} and income of system S_i will decrease on this, where R_{ij} – RV with d.f. $F_{2ij}(x)$, $i,j=\overline{1,n}$; e) change of system S_i state won't occur with probability $1-(\lambda p_{0i}+\mu_i)\Delta t+o(\Delta t)$ at the time interval Δt.
It's evident that $r_{ji}=R_{ji}$ with probability 1, $i,j=\overline{1,n}$, i.e.

$$F_{1ij}(x)=F_{2ij}(x),\ i,j=\overline{1,n} \tag{1}$$

Besides system S_i increases its income on value $r_i\Delta t$ during every small time interval Δt due to interest on money that are in it; let us suppose that r_i is RV with d.f. $F_i(x), i=\overline{1,n}$. We will suppose also that RV r_{ji}, R_{ij}, r_{0i}, R_{i0}, r_i are independent in pairs, $i,j=\overline{1,n}$.

Let $d_{il}(\Delta t)$ is income change of i-th QS at the l-th time interval of length Δt. Then from all that was saying before it follows

$$d_{il}(\Delta t)=\begin{cases} r_{0i}+r_i\Delta t \quad \text{with probability } \lambda p_{0i}\Delta t+o(\Delta t),\\ -R_{i0}+r_i\Delta t \quad \text{with probability } \mu_i p_{i0}\Delta t+o(\Delta t),\\ r_{ji}+r_i\Delta t \quad \text{with probability } \mu_j p_{ji}\Delta t+o(\Delta t), j=\overline{1,n}, j\neq i,\\ -R_{ij}+r_i\Delta t \quad \text{with probability } \mu_i p_{ij}\Delta t+o(\Delta t), j=\overline{1,n}, j\neq i,\\ r_i\Delta t \quad \text{with probability } 1-(\lambda p_{0i}+\mu_i)\Delta t+o(\Delta t). \end{cases} \tag{2}$$

Income of system S_i equals

$$V_i(t)=v_{i0}+\sum_{l=1}^{m} d_{il}(\Delta t) \tag{3}$$

Let us introduce definition for expectation values (e.v.) correspondently:

$$M\{r_{ji}\}=\int_0^\infty x dF_{1ji}(x)=a_{ji},\ M\{R_{ij}\}=\int_0^\infty x dF_{2ij}(x)=b_{ij},\ M\{r_i\}=\int_0^\infty x dF_i(x)=c_i$$

$$M\{r_{0i}\}=\int_0^\infty x dF_{0i}(x)=a_{0i},\ M\{R_{i0}\}=\int_0^\infty x dF_{i0}(x)=b_{i0},\ i,j=\overline{1,n} \tag{4}$$

in view of (1)

$$a_{ji}=b_{ji},\ i,j=\overline{1,n} \tag{5}$$

Let us derive expression for e.v. of income of system S_i at the moment t. According to formula for e.v. of discrete RQ

$$M\{d_{il}(\Delta t)\} = \left(\lambda a_{0i} p_{0i} + \sum_{j=1}^{n} \mu_j a_{ji} p_{ji} - \mu_i \sum_{j=0}^{n} b_{ij} p_{ij} + c_i \right) \Delta t + o(\Delta t), \ i = \overline{1,n} \quad (6)$$

As it follows from (3) taking into account $m\Delta t = t$, when $\Delta t \to 0$

$$v_i(t) = M\{V_i(t)\} = v_{i0} + \left[c_i + \lambda a_{0i} p_{0i} + \sum_{j=1}^{n} \mu_j a_{ji} p_{ji} - \mu_i \sum_{j=0}^{n} b_{ij} p_{ij} \right] t \quad (7)$$

For expected income for whole network considering (5), we have:

$$M\{W(t)\} = \sum_{i=1}^{n} v_i(t) = \sum_{i=1}^{n} \left[v_{i0} + \left(c_i + \lambda a_{0i} p_{0i} - \mu_i b_{i0} p_{i0} \right) t \right] \quad (8)$$

Let us notice that common expected network's income doesn't depend on r_{ij}, R_{ij}, $i, j = \overline{1,n}$, as that incomes absorb each other (if message transmits from one network's system to another then income of the last one increases on some value and income of the first QS decreases on the same value).

2. Variations of network systems incomes

From expressions (2), (3) it follows that

$$M^2\{V_i(t) - v_{i0}\} = \left[c_i + \lambda a_{0i} p_{0i} - \mu_i b_{i0} p_{i0} + \sum_{j=1}^{n} \left(\mu_j a_{ji} p_{ji} - \mu_i b_{ij} p_{ij} \right) \right]^2 t^2$$

Let us introduce denotations:

$$M\{r_{0i}{}^2\} = a_{20i}, \quad M\{R_{i0}{}^2\} = b_{2i0}, \ M\{r_{ji}{}^2\} = a_{2ji}, \quad M\{R_{ij}{}^2\} = b_{2ij}, \quad i, j = \overline{1,n} \quad (9)$$

Let us consider expression

$$M(V_i(t) - v_{i0})^2 = M\left(v_{i0} + \sum_{l=1}^{m} d_{il}(\Delta t) - v_{i0} \right)^2 = M\left(\sum_{l=1}^{m} d_{il}(\Delta t) \right)^2 =$$

$$= \sum_{l=1}^{m} Md_{il}^2(\Delta t) + \sum_{l=1}^{m} \sum_{\substack{j=1 \\ j \neq l}}^{m} M\left(d_{il}(\Delta t) d_{ij}(\Delta t) \right), i = \overline{1,n} \quad (10)$$

Taking into account (2)-(9), we have:

$$M\left\{d_{il}^{\,2}(\Delta t)\right\}=M\left[\left(r_{0i}^2+2r_{0i}r_i\ \Delta t+r_i^2(\Delta t)^2\right)\lambda p_{0i}\Delta t\right]+$$

$$+M\left[\left(R_{i0}^2-2R_{i0}r_0\ \Delta t+r_i^2(\Delta t)^2\right)\mu_i p_{ij}\Delta t\right]+\sum_{\substack{j=1\\ j\neq i}}^{n}M\left[\left(r_{ji}^2+2r_{ji}r_i\ \Delta t+\right.\right.$$

$$\left.\left.+r_i^2(\Delta t)^2\right)\mu_j p_{ji}\Delta t\right]+\sum_{\substack{j=1\\ j\neq i}}^{n}M\left[\left(R_{ij}^2-2R_{ij}r_i\ \Delta t+r_i^2(\Delta t)^2\right)\mu_i p_{ij}\Delta t\right]+$$

$$+Mr_i^2(\Delta t)^2\left[1-\left(\lambda p_{0i}+\mu_i\right)\Delta t\right]+o(\Delta t)=$$

$$=\left[\lambda a_{20i}p_{0i}+\mu_i b_{2i0}p_{i0}+\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\mu_j a_{2ji}p_{ji}+\mu_i b_{2ij}p_{ij}\right)\right]\Delta t+o(\Delta t)\,,\ i=\overline{1,n} \qquad (11)$$

Besides from independence of $d_{il}(\Delta t)$ and $d_{ij}(\Delta t)$, $j\neq i$, considering (2) it follows that $M\left(d_{il}(\Delta t)d_{iq}(\Delta t)\right)=o(\Delta t)^2$. So as it's fallowed from (10), (11) and $m\Delta t=t$, for variation of income of i-th QS when $\Delta t\to 0$ we obtain the following expression

$$DV_i(t)=D\left(V_i(t)-v_{i0}\right)=M\left\{\left(V_i(t)-v_{i0}\right)^2\right\}-M^2\left\{\left(V_i(t)-v_{i0}\right)\right\}=$$

$$=\left[\lambda a_{20i}p_{0i}+\mu_i b_{2i0}p_{i0}+\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\mu_j a_{2ji}p_{ji}+\mu_i b_{2ij}p_{ij}\right)\right]t+$$

$$+\left[\mu_i b_{i0}p_{i0}-\lambda a_{0i}p_{0i}-c_i+\sum_{j=1}^{n}\left(\mu_i a_{ij}p_{ij}-\mu_j a_{ji}p_{ji}\right)\right]t^2,\ i=\overline{1,n} \qquad (12)$$

3. Expected incomes of Jackson HM-network systems

Let us consider now the case when service rate of messages $\mu_i(k_i)$ in system S_i depends on number of messages that are present in it, $i=\overline{1,n}$. So we take off limitations that network must functioning in the term of high capacity. In our case expression (2) will look as

$$d_{il}(\Delta t)=\begin{cases} r_{0i}+r_i\Delta t \quad \text{with probability } \lambda p_{0i}\Delta t+o(\Delta t), \\ -R_{i0}+r_i\Delta t \quad \text{with probability } \mu_i(k_i(l))u(k_i(l))p_{i0}\Delta t+o(\Delta t), \\ r_{ji}+r_i\Delta t \quad \text{with probability } \mu_j(k_j(l))u(k_j(l))p_{ji}\Delta t+o(\Delta t), \\ -R_{ij}+r_i\Delta t \quad \text{with probability } \mu_i(k_i(l))u(k_i(l))p_{ij}\Delta t+o(\Delta t), \\ r_i\Delta t \quad \text{with probability } 1-\left(\lambda p_{0i}+\mu_i(k_i(l))u(k_i(l))\right)\Delta t+o(\Delta t), \end{cases}$$

$$j=\overline{1,n}, j\neq i \tag{13}$$

where $k_i(l)$ is number of messages in the i-th QS at the l-th time interval, $u(x)=\begin{cases}1, x>0,\\ 0, x\le 0\end{cases}$ - Heavyside function.

Let us find expression for expected income of system S_i at the moment *t*. When realization of process $k(t)$ is fixed, according to (13) and denotation (5), we can write:

$$M\{d_{il}(\Delta t)/k(t)\}=\left[\lambda a_{0i}p_{0i}+\sum_{j=1}^{n}a_{ji}p_{ji}\mu_j(k_j(l))u(k_j(l))-\right.$$

$$\left.-\mu_i(k_i(l))u(k_i(l))\sum_{j=0}^{n}b_{ij}p_{ij}\right]\Delta t+c_i\Delta t+o(\Delta t),\ i=\overline{1,n} \tag{14}$$

Then taking into account $m\Delta t=t$ and (3) we obtain

$$M\{V_i(t)/k(t)\}=\sum_{l=1}^{m}M\{d_{il}(\Delta t)/k(t)\}=\left(\lambda a_{0i}p_{0i}+c_i\right)t+$$

$$+\sum_{j=1}^{n}a_{ji}p_{ji}\sum_{l=1}^{m}\mu_j(k_j(l))u(k_j(l))\Delta t-$$

$$-\sum_{j=0}^{n}b_{ij}p_{ij}\sum_{l=1}^{m}\mu_i(k_i(l))u(k_i(l))\Delta t+o(\Delta t),\ i=\overline{1,n} \tag{15}$$

When $m \to \infty$, $\Delta t \to 0$

$$\sum_{l=1}^{m} \mu_j(k_j(l))u(k_j(l))\Delta t \xrightarrow[\Delta t \to 0]{} \int_0^t \mu_j(k_j(s))u(k_j(s))ds\,, \quad j=\overline{1,n} \qquad (16)$$

so

$$M\{V_i(t)/k(t)\} = v_{i0} + (c_i + \lambda a_{0i} p_{0i})t + \sum_{j=1}^{n} a_{ji} p_{ji} \int_0^t \mu_j(k_j(s))u(k_j(s))ds -$$

$$-\int_0^t \mu_i(k_i(s))u(k_i(s))ds \sum_{j=0}^{n} b_{ij} p_{ij}\,, \quad i=\overline{1,n} \qquad (17)$$

Making average by $k(t)$ and taking into account condition of normalization $\sum_k P(k(t)=k)=1$ for expected income of system S_i we will have

$$v_i(t) = M\{V_i(t)\} = v_{i0} + \sum_k P(k(t)=k) M\{V_i(t)/k(t)\} =$$

$$= v_{i0} + (c_i + \lambda a_{0i} p_{0i})t + \sum_k P(k(t)=k) \left[\sum_{j=1}^{n} a_{ji} p_{ji} \int_0^t \mu_j(k_j(s))u(k_j(s))ds - \right.$$

$$\left. -\int_0^t \mu_i(k_i(s))u(k_i(s))ds \sum_{j=0}^{n} b_{ij} p_{ij} \right], \quad i=\overline{1,n} \qquad (18)$$

Let system S_i consists of m_i identical service lines, the service rate of messages in each line equals μ_i, $i=\overline{1,n}$. In this case

$$\mu_i(k_i(s))u(k_i(s)) = \begin{cases} \mu_i k_i(s), & k_i(s) \le m_i, \\ \mu_i m_i, & k_i(s) > m_i, \end{cases} = \mu_i \min(k_i(s), m_i), \quad i=\overline{1,n} \qquad (19)$$

Then from (18) it follows

$$v_i(t) = M\{V_i(t)\} = v_{i0} + (c_i + \lambda a_{0i} p_{0i})t + \sum_k P(k(t)=k) \times$$

$$\times \left[\sum_{j=1}^{n} \mu_j a_{ji} p_{ji} \int_0^t \min(k_j(s), m_j)ds - \mu_i \int_0^t \min(k_i(s), m_i)ds \sum_{j=0}^{n} b_{ij} p_{ij} \right], \quad i=\overline{1,n} \qquad (20)$$

It is natural to suppose that average of expression $\min(k_j(s), m_j)$ gives $\min(N_j(s), m_j)$, i.e.

$$M\min(k_j(s), m_j) = \min(N_j(s), m_j), \ i = \overline{1,n} \tag{21}$$

where $N_j(s) = M\{k_j(s)\}$ is average number of messages (waiting and servicing) in system S_i at time interval $[t_0, t_0 + s]$, $i = \overline{1,n}$. Therefore we receive from (20) the following relation

$$v_i(t) = v_{i0} + (c_i + \lambda a_{0i} p_{0i})t + \sum_{j=1}^{n} \mu_j a_{ji} p_{ji} \int_0^t \min(N_j(s), m_j) ds -$$

$$- \mu_i \int_0^t \min(N_i(s), m_i) ds \sum_{j=0}^{n} b_{ij} p_{ij}, \ i = \overline{1,n} \tag{22}$$

As

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mu_j a_{ji} p_{ji} \int_0^t \min(N_j(s), m_j) ds = \sum_{i=1}^{n} \mu_i \int_0^t \min(N_i(s), m_i) ds \sum_{j=1}^{n} b_{ij} p_{ij}$$

so expected income of whole network equals

$$M\{W(t)\} = \sum_{i=1}^{n} \left[v_{i0} + (c_i + \lambda a_{0i} p_{0i})t - \mu_i b_{i0} p_{i0} \int_0^t \min(N_i(s), m_i) ds \right]$$

For finding of $N_i(t)$, $i = \overline{1,n}$, it can be applied recurrence by the time moments method of analysis of average values for open QN [10].

4. Variations of incomes in Jackson HM-network

Taking into account (3), expression (22) can be rewritten as

$$v_i(t) = v_{i0} + (c_i + \lambda a_{0i} p_{0i})t - \mu_i b_{i0} p_{i0} \int_0^t \min(N_i(s), m_i) ds +$$

$$+ \sum_{j=1}^{n} \left[\mu_j a_{ji} p_{ji} \int_0^t \min(N_j(s), m_j) ds - \mu_i a_{ij} p_{ij} \int_0^t \min(N_i(s), m_i) ds \right], \ i = \overline{1,n} \tag{23}$$

From (13), (19), (9) it follows (similar as we found (11)):

$$M\left\{d_{il}{}^{2}(\Delta t)/k(t)\right\}=\{\lambda a_{20i}p_{0i}+\mu_i b_{2i0}p_{i0}\min(k_i(l),m_i)+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j a_{2ji}p_{ji}\min(k_j(l),m_j)+\mu_i b_{2ij}p_{ij}\min(k_i(l),m_i)\right]\Delta t+o(\Delta t),\ i=\overline{1,n} \qquad (24)$$

Besides taking into account independence of $d_{il}(\Delta t)$, $d_{iq}(\Delta t)$, $l\neq q$, and (13) we can write

$$M\left(d_{il}(\Delta t)d_{iq}(\Delta t)/k(t)\right)=o(\Delta t)^2 \qquad (25)$$

Then making limit transition when $\Delta t\to 0$, from (24), (25) and consider that $m\Delta t=t$, we have

$$M\left\{(V_i(t)-v_{i0})^2\left|k(t)\right.\right\}=\sum_{l=1}^{m}M\left\{d_{il}{}^{2}(\Delta t)/k(t)\right\}+\sum_{l=1}^{m}\sum_{\substack{q=1\\ q\neq l}}^{m}M\left\{d_{il}(\Delta t)d_{iq}(\Delta t)/k(t)\right\}=$$

$$=\lambda a_{20i}p_{0i}t+\mu_i b_{2i0}p_{i0}\int_0^t\min(k_i(s),m_i)ds+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j a_{2ji}p_{ji}\int_0^t\min(k_j(s),m_j)ds+\mu_i b_{2ij}p_{ij}\int_0^t\min(k_i(s),m_i)ds\right],\ i=\overline{1,n}$$

Making average of this expression by $k(t)$ we will have

$$M\left\{(V_i(t)-v_{i0})^2\right\}=\lambda a_{20i}p_{0i}t+\mu_i b_{2i0}p_{i0}\int_0^t\min(N_i(s),m_i)ds+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j a_{2ji}p_{ji}\int_0^t\min(N_j(s),m_j)+\mu_i b_{2ij}p_{ij}\int_0^t\min(N_i(s),m_i)\right],\ i=\overline{1,n} \qquad (26)$$

Let us find expression for $M^2\{(V_i(t)-v_{i0})\}$ using (23):

$$M^2\{(V_i(t)-v_{i0})\}=\left\{(c_i+\lambda a_{0i}p_{0i})t-\mu_i b_{i0}p_{i0}\int_0^t\min(N_i(s),m_i)ds+\right.$$

$$+\sum_{j=1}^{n}\left[\mu_j a_{ji} p_{ji}\int_0^t \min(N_j(s),m_j)ds-\mu_i a_{ij} p_{ij}\int_0^t \min(N_i(s),m_i)ds\right]\Bigg\}^2, \quad i=\overline{1,n} \qquad (27)$$

Expression for variation of income of the i-th network's QS can be easily found using formula $DV_i(t)=M\{(V_i(t)-v_{i0})^2\}-M^2\{(V_i(t)-v_{i0})\}$ and relations (26), (27).

5. Some optimization problems for network incomes

Let us to consider open QN with incomes that was described in the previous sections. Let us formulate two optimization problems that are associated with income's maximization of the whole network or central QS for example

$$\begin{cases} W_1(T,m)=W(T,m_1,...,m_n)=\dfrac{1}{T-t_0}\displaystyle\int_{t_0}^{T}\sum_{i=1}^{n}\left(MV_i(t)-d_iN_i(t)-E_im_i\right)dt\to\max_{m_1,m_2,...,m_n} \\ m_i\le a_i,\ i=\overline{1,n} \end{cases} \qquad (28)$$

$$\begin{cases} W_2(T,m)=W_2(T,m_1,...,m_n)=\dfrac{1}{T-t_0}\displaystyle\int_{t_0}^{T}\left(v_n(t)-d_nN_n(t)-E_nm_n\right)dt\to\max_{m_1,m_2,...,m_n} \\ m_i\le a_i,\ i=\overline{1,n} \end{cases} \qquad (29)$$

where a_i - set numbers that present range in which number of service lines in the i-th QS can change; E_i - coasts for keeping of one service line; d_i - coasts for keeping of one message in the i-th system, $i=\overline{1,n}$. From relation (20) it is evident that (28) and (29) can be reduced to the next problems

$$\begin{cases} \dfrac{1}{T-t_0}\displaystyle\int_{t_0}^{T}\sum_{i=1}^{n}\left(\mu_i b_{i0} p_{i0}\int_{t_0}^{t}\min(N_i(s),m_i)ds+d_iN_i(t)+E_im_i\right)dt\to\min_{m_1,m_2,...,m_r} \\ m_i\le a_i,\ i=\overline{1,n} \end{cases} \qquad (30)$$

$$\begin{cases} \dfrac{1}{T-t_0}\displaystyle\int_{t_0}^{T}\left[\sum_{j=1}^{n}\mu_j a_{jn} p_{jn}\int_{t_0}^{t}\min(N_j(s),m_j)ds-\mu_n\int_{t_0}^{t}\min(N_n(s),m_n)ds\sum_{j=0}^{n}b_{nj}p_{nj}-\right. \\ \qquad\left.-\left(d_nN_n(t)+E_nm_n\right)\right]dt\to\max_{m_1,m_2,...,m_n} \\ m_i\le a_i,\ i=\overline{1,n} \end{cases} \qquad (31)$$

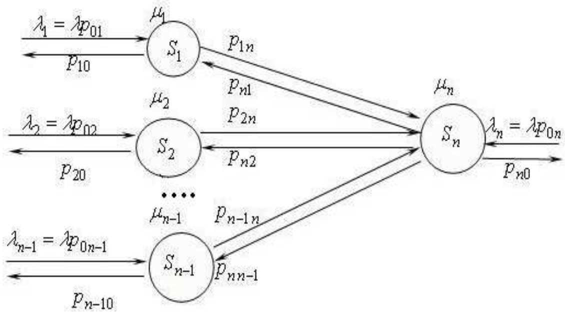

Fig. 1. Topology of HM-network

Integrals in problems (30) and (31) can be found by applying method of trapeziums two times sequentially.

6. Modelling example

Let us consider open HM-network with central system, (Fig. 1). Network is discribed with the next parametrs: count of QS in network $n = 20$, $\lambda = 15$; the starting vector of service line's count in network systems consists of elements $m_i = 1$, $N_i(0) = 5$, $a_i = 10$, $i = \overline{1,20}$; probobilities of message transitions between network's systems are equal $p_{0i} = 0.063q^{i-1}$, $q = 0.975$, $i = \overline{1,20}$, (they are found with a help of formula of geometric series and condition $\sum_{i=1}^{20} p_{0i} = 1$), $p_{i0} = p_{i20} = 0.5$, $i = \overline{1,19}$, $p_{20\,0} = 0.5$, $p_{20\,i} = 0.026$, $i = \overline{1,19}$, other $p_{ij} = 0$, $i, j = \overline{1,19}$. In this case vector of message arrival rates in each system consists of components $\lambda\ p_{0i}$ and equals (1.219, 1.195, 1.172, 1.149, 1.128, 1.106, 1.086, 1.065, 1.045, 1.026, 1.01, 0.989, 0.971, 0.954, 0.937, 0.92, 0.904, 0.888, 0.873, 10.401). Service times of messages in each system's line are distributed with exponential law with parametrs μ_i that are equal (2.119, 2.695, 2.07, 1.349, 1.728, 2.606, 1.386, 1.165, 2.045, 2.426, 1.707, 1.989, 2.371, 1.754, 2.037, 2.32, 1.204, 1.588, 0.973, 11.8); vector with components d_i looks like (0.25, 0.28, 0.26, 0.5, 0.8, 0.3, 0.17, 0.5, 0.10, 0.7, 0.10, 0.11, 0.17, 0.18, 0.10, 0.80, 0.2, 0.51, 0.4, 0.14) and vector with components E_i - (0.10, 0.19, 0.17, 0.8, 0.13, 0.3, 0.5, 0.2, 0.11, 0.4, 0.11, 0.16, 0.7, 0.18, 0.13, 0.15, 0.15 ,0.20, 0.12, 0.10). Let also e.v. are equal $a_{j20} = 27$, $j = 1,18,19,20$, $a_{j20} = 31$, $j = 2,3,4,5,9,10,11$, $a_{j\,20} = 19$, $j = 8,15,16,17$, $a_{j20} = 21$, $j = 6,7,12,13,14$; $b_{j0} = 11$, $j = 11,20$, $b_{j0} = 13$, $j = 5,7,9,16$,

$b_{j0} = 15,\ j = 2,6,13,15,18,\quad b_{j0} = 17,\ j = 3,4,14,19,\quad b_{j0} = 19,\quad j = 1,8,10,12,17;$ $b_{nj} = 12,\ j = 3,14,15,19,\quad b_{nj} = 14,\ j = 5,20,\quad b_{nj} = 16,\quad j = 2,4,9,10,18,$ $b_{nj} = 18,\ j = 7,13,17,\ b_{nj} = 11,\ j = 1,6,8,11,12,16;\ t_0 = 0,\ T = 50.$

Optimization problems (30) and (31) were solved by method of full search. Solution of problem (30) is $m_i^* = 1,\ i = \overline{1,19},\ m_{20}^* = 2$ and value of optimization criteria in this case is $W_1(50, m^*) = 9.138$. And solution of optimization problem (31) is $m_i^* = 1,\ i = \overline{1,19},\ i \neq 4,\ m_4^* = 2,\ m_{20}^* = 3,$ and value of optimization criteria in this case is $W_2(50, m_i^*) = -4.94$.

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