MULTIOBJECTIVE IDENTIFICATION OF BOUNDARY TEMPERATURE USING ENERGY MINIMIZATION METHOD COUPLED WITH BOUNDARY ELEMENT METHOD

Marek Jasiński

Silesian University of Technology, Gliwice, Poland, marek.jasinski@polsl.pl

Abstract. The inverse problem consisting in an identification of boundary temperature is discussed. As the identification method the Pareto approach with two criteria connected with domain and boundary has been used. In order to solve the problem the energy minimization method connected with boundary element method for steady state problem has been employed. The theoretical considerations are supplemented by the examples of computations verifying the correctness of the algorithm proposed.

1. Governing equations

Let us consider 2D problem of temperature identification on boundary Γ_1 [1, 2]:

$$\begin{cases} x \in \Omega: & \dfrac{\partial^2 T}{\partial x_1^2} + \dfrac{\partial^2 T}{\partial x_2^2} = 0 \\ x \in \Gamma_1: & T(x) = ? \\ x \in \Gamma_2: & T(x) = T_b \\ x \in \Gamma_3: & q(x) = -\lambda \dfrac{\partial T}{\partial n} = q_b \\ \xi^i \in \Omega: & T_d(\xi^i) - \text{known},\ i=1\ldots M \end{cases} \tag{1}$$

where λ [W/(mK)] is thermal conductivity, T denotes temperature, T_b, q_b are the given boundary temperature and heat flux, $T_d(\xi^i)$ are the known temperatures at the internal points ξ^i from the domain considered.

2. Energy minimization method coupled with the BEM

The boundary integral equation for problem (1) is of the form [4]:

$$B(\xi)T(\xi) + \int_\Gamma q(x)T^*(\xi, x)\,\mathrm{d}\Gamma = \int_\Gamma T(x)q^*(\xi, x)\,\mathrm{d}\Gamma \tag{2}$$

where $\xi \in \Gamma$ is the observation point, $T^*(\xi, x)$ is the fundamental solution, while $q^* = -\lambda \partial T^*(\xi, x)/\partial n$ and $B(\xi) \in (0,1)$.

In numerical realization, the boundary Γ is divided into N constant boundary elements Γ_j [1,5]. Additionally, we assume that N_1 nodes belong to the boundary Γ_1, the nodes $N_1 + 1,\ldots, N_2$ belong to Γ_2, $N_2 + 1,\ldots, N$ belong to Γ_3 (Fig. 1). The integrals in equation (2) are substituted by sum of integrals and then, taking into account the boundary conditions (1), one obtains

$$\xi^i \in \Gamma: \ \sum_{j=1}^{N_2} G_{ij} q_j + \sum_{j=N_2+1}^{N} G_{ij} q_b = \sum_{j=1}^{N_1} H_{ij} T_j + \sum_{j=N_1+1}^{N_2} H_{ij} T_b + \sum_{j=N_2+1}^{N} H_{ij} T_j \tag{3}$$

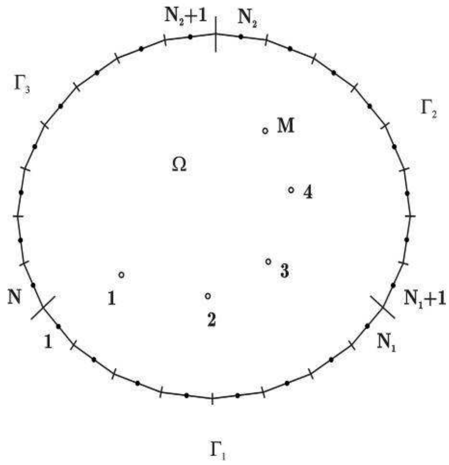

Fig. 1. The domain considered

where [4]

$$\xi^i \in \Gamma: \quad G_{ij} = \int_{\Gamma_j} T^*(\xi^i, x)\,\mathrm{d}\Gamma_j, \quad H_{ij} = \begin{cases} \int_{\Gamma_j} q^*(\xi^i, x)\,\mathrm{d}\Gamma_j, & i \neq j \\ -0.5, & i = j \end{cases} \tag{4}$$

The well - ordered system of equations has the form

$$\mathbf{Y} = \mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{P} = \mathbf{UP} \tag{5}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} G_{11} & \ldots & G_{1N_2} & -H_{1N_2+1} & \ldots & -H_{1N} \\ & \ldots & & & \ldots & \\ G_{N1} & \ldots & G_{NN_2} & -H_{NN_2+1} & \ldots & -H_{NN} \end{bmatrix} \tag{6}$$

$$\mathbf{A}_2 = \begin{bmatrix} H_{11} & \dots & H_{1N_2} & -G_{1N_2+1} & \dots & -G_{1N} \\ & \dots & & & \dots & \\ H_{N1} & \dots & H_{NN_2} & -G_{NN_2+1} & \dots & -G_{NN} \end{bmatrix} \tag{7}$$

$$\mathbf{Y} = \begin{bmatrix} q_1 & \dots & q_{N_2} & T_{N_2+1} & \dots & T_N \end{bmatrix}^{\mathrm{T}} \tag{8}$$

$$\mathbf{P} = \begin{bmatrix} T_1 & \dots & T_{N_1} & T_b & \dots & T_b & q_b & \dots & q_b \end{bmatrix}^{\mathrm{T}} \tag{9}$$

The temperatures at internal nodes ξ^i are calculated using the formula [1, 4]

$$\begin{gathered} T(\xi^i) = \sum_{j=1}^{N_1} H_{ij}^w T_j + \sum_{j=N_1+1}^{N_2} H_{ij}^w P_j + \sum_{j=N_2+1}^{N} H_{ij}^w T_j - \\ -\sum_{j=1}^{N_1} G_{ij}^w q_j - \sum_{j=N_1+1}^{N_2} G_{ij}^w q_j - \sum_{j=N_2+1}^{N} G_{ij}^w P_j \end{gathered} \tag{10}$$

From the system of equations (5) results that

$$q_j = \sum_{k=1}^{N_1} U_{jk} T_k + \sum_{k=N_1+1}^{N} U_{jk} P_k, \quad j = 1, 2, \dots, N_2 \tag{11}$$

$$T_j = \sum_{k=1}^{N_1} U_{jk} T_k + \sum_{k=N_1+1}^{N} U_{jk} P_k, \quad j = N_2+1, N_2+2, \dots, N \tag{12}$$

Putting (11) and (12) into (10) one has

$$\begin{gathered} T(\xi^i) = \sum_{j=1}^{N_1} \left[H_{ij}^w - \sum_{k=1}^{N_2} G_{ik}^w U_{kj} + \sum_{k=N_2+1}^{N} H_{ik}^w U_{kj} \right] T_j + \\ + \sum_{j=N_1+1}^{N} \left[-\sum_{k=1}^{N_2} G_{ik}^w U_{kj} + \sum_{k=N_2+1}^{N} H_{ik}^w U_{kj} \right] P_j + \sum_{j=N_1+1}^{N_2} H_{ij}^w P_j - \sum_{j=N_2+1}^{N} G_{ij}^w P_j \end{gathered} \tag{13}$$

or

$$T(\xi^i) = \sum_{j=1}^{N_1} W_{ij} T_j + Z_i \tag{14}$$

where

$$W_{ij} = H_{ij}^w - \sum_{k=1}^{N_2} G_{ik}^w U_{kj} + \sum_{k=N_2+1}^{N} H_{ik}^w U_{kj} \tag{15}$$

$$Z_i = \sum_{j=N_1+1}^{N} \left(-\sum_{k=1}^{N_2} G_{ik}^w U_{kj} + \sum_{k=N_2+1}^{N} H_{ik}^w U_{kj} \right) P_j + \sum_{j=N_1+1}^{N_2} H_{ij}^w P_j - \sum_{j=N_2+1}^{N} G_{ij}^w P_j \tag{16}$$

In order to solve the problem considered, the energy minimization method is applied which resolves itself into seek of minimum of some functional with the following restrictions (c.f. equation (14)) [1-3]

$$\left|T(\xi^i)-T_d(\xi^i)\right| = \left|\sum_{j=1}^{N_1} W_{ij}T_j - F(\xi^i)\right| \le \varepsilon, \quad i=1,...,M \tag{17}$$

where

$$F(\xi^i) = T_d(\xi^i) - Z_i \tag{18}$$

3. Pareto approach

Multiobjective optimization (MOO) problem can be expressed as searching for the vector **x** from a set of admissible solution which minimizes the vector of k objective functions [5]

$$\mathbf{J}(\mathbf{x}) = \left[J_1(\mathbf{x}), \quad J_2(\mathbf{x}), \quad ... \quad J_k(\mathbf{x})\right]^{\mathrm{T}} \tag{19}$$

Vector **x** must satisfy the p inequality constrains and r equality constrains

$$\begin{aligned} f_i(x) &\ge 0 \quad i=1,2,...,p \\ f_i(x) &= 0 \quad i=p+1, p+2,..., p+r \end{aligned} \tag{20}$$

According to the Pareto optimality concept, a point $\mathbf{x}^*$ is Pareto - optimal in the minimization problems if and only if there does not exist another point **x** such that

$$\mathbf{J}(\mathbf{x}) \le \mathbf{J}(\mathbf{x}^*) \tag{21}$$

and additionally at last one

$$J_i(\mathbf{x}) < J_i(\mathbf{x}^*) \tag{22}$$

The set of Pareto optimal solutions is called the Pareto front. In other words, it is a set of so-called non-dominated or efficient solutions.

A standard technique for identification Pareto - optimal points is minimization of weighted sums of functions called metacriterion [5]

$$K(\mathbf{x}) = \sum_{i=1}^{k} w_i J_i(\mathbf{x}) \tag{23}$$

where

$$\sum_{i=1}^{k} w_i = 1 \tag{24}$$

To obtain a set of Pareto front points, the minimization of set of functions (23) with different weights is demanded.

In current analysis, the bi-objective problem is considered with functional connected with boundary of the domain analyzed [3]

$$J_1 = -\frac{1}{2\lambda}\int_{\Gamma} T(x)q(x)\,\mathrm{d}\Gamma \tag{25}$$

and connected with the interior of the domain

$$J_2 = \sum_{i=1}^{M}\left[T(\xi^i) - T_d(\xi^i)\right]^2 \tag{26}$$

Both functionals are connected with minimization energy method through appropriate equations: (8), (9) for objective function J_1, and (14), (18) for functional J_2.

4. Examples of computations

The square of dimensions 0.1×0.1 m has been considered. The thermal conductivity λ = 1 W/mK.

The boundary has been divided into 40 constant boundary elements. In order to solve the inverse problem, it is assumed that the values of temperature are known at 12 selected points from interior of the domain - Figure 2.

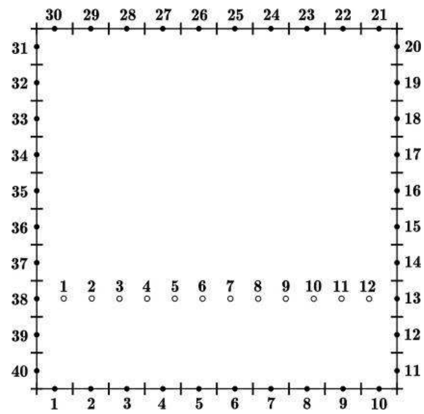

Fig. 2. Discretization and position of internal points

Temperatures at internal points have been obtained from the direct problem (1) solution for the boundary conditions collected in Table 1. The parameter ε = 0.5, the distance between bottom side and sensors is equal 0.025 m.

Table 1

Boundary conditions

	1st variant	2nd variant
Left side	Tb = 50°C	qb = 0
Top side	Tb = 100°C	qb = 0
Right side	Tb = 100°C	Tb = 100°C
Bottom side	Tb = 50°C	Tb = 50°C

Multiobjective optimization for several weights combination has been made. In Figure 3 the Pareto front obtained for 1st variant of boundary conditions is presented.

Descriptions on the axes are defined as

$$\overset{*}{J}_1 = J_1 - J_{1k} \tag{27}$$

and

$$\overset{*}{J}_2 = J_2 - J_{2k} \tag{28}$$

where J_1 and J_2 correspond to the values of functional from single - objective optimization (or to weights combinations $w_1 = 1$, $w_2 = 0$ for J_1 and $w_1 = 0$, $w_2 = 1$ for J_2), while J_{1k} and J_{2k} are the Pareto optimal point co-ordinates obtained for k th weights combination.

In Table 2 the results of computations for selected weights combinations for 1st variant of boundary conditions are shown.

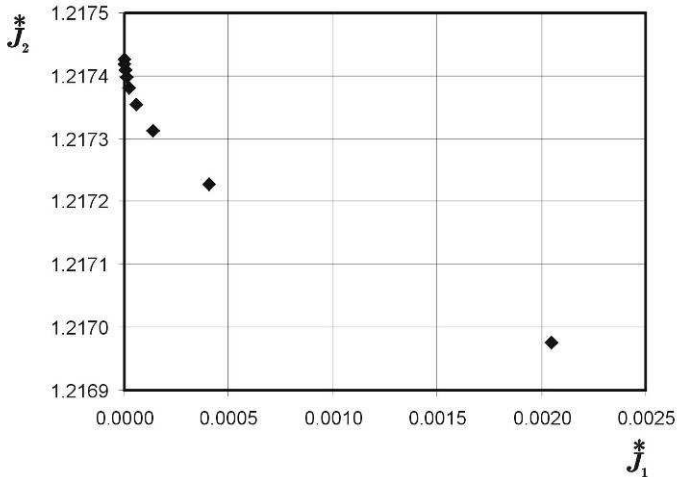

Fig. 3. Pareto front (1st variant)

Table 2

Solution of inverse problem (1st variant)

Node	w1 = 0.9 w2 = 0.1	w1 = 0.7 w2 = 0.3	w1 = 0.5 w2 = 0.5	w1 = 0.3 w2 = 0.7	w1 = 0.1 w2 = 0.7
1	50.4119	50.4119	50.4118	50.4116	50.4106
2	51.1892	51.1890	51.1888	51.1883	51.1859
3	51.5447	51.5446	51.5444	51.5439	51.5416
4	51.2549	51.2548	51.2547	51.2545	51.2536
5	50.0737	50.0738	50.0738	50.0739	50.0743
6	47.8412	47.8413	47.8414	47.8415	47.8423
7	45.0298	45.0298	45.0298	45.0299	45.0303
8	44.5127	44.5127	44.5127	44.5127	44.5127
9	53.8725	53.8725	53.8725	53.8725	53.8724
10	80.6899	80.6899	80.6899	80.6899	80.6899
K	–984 312	–763 165	–542 019	–320 872	–99 725

The results of computations for 2^{nd} variant of boundary conditions are presented in Figure 4 and Table 3. Figure 4 illustrates the set of Pareto - optimal points obtained for such kind of boundary conditions, while in Table 3 the values of identified temperatures for different weights are shown.

Table 3

Solution of inverse problem (2nd variant)

Node	w1 = 0.9 w2 = 0.1	w1 = 0.7 w2 = 0.3	w1 = 0.5 w2 = 0.5	w1 = 0.3 w2 = 0.7	w1 = 0.1 w2 = 0.7
1	50.96078	50.96084	50.96095	50.96122	50.96253
2	51.20679	51.20678	51.20676	51.20670	51.20640
3	51.29075	51.29068	51.29055	51.29025	51.28876
4	50.93627	50.93621	50.93612	50.93591	50.93483
5	49.83781	49.83782	49.83782	49.83784	49.83791
6	47.75828	47.75830	47.75835	47.75847	47.75904
7	45.03993	45.03995	45.03998	45.04004	45.04034
8	44.51270	44.51270	44.51270	44.51269	44.51265
9	53.84629	53.84629	53.84628	53.84626	53.84618
10	80.66937	80.66937	80.66937	80.66938	80.66939
K	–428 180	–327 366	–226 552	–125 738	–24 925

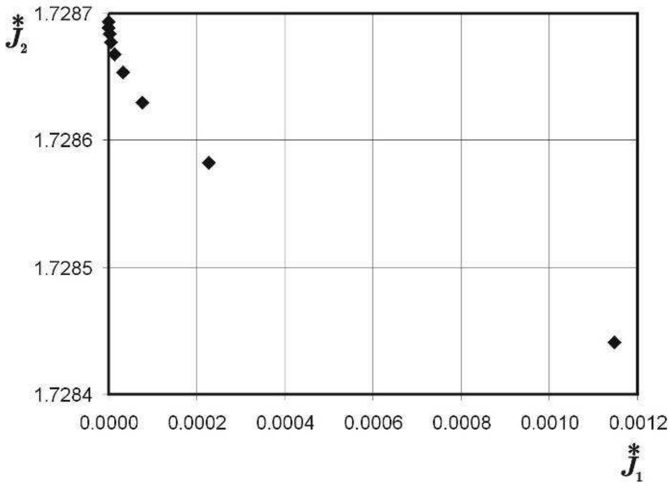

Fig. 4. Pareto front (2nd variant)

5. Final remarks

In both variants of boundary conditions the differences between temperatures identified for selected set of weights are very small, however the differences in optimal values of metacriterion are clear visible (c. f. Tables 2 and 3). It should be pointed out that most of Pareto - optimal points are located closer to the optimal values of functional J_1 corresponding to Pareto optimal point co-ordinations for $w_1 = 1$ and $w_2 = 0$ (or to single - objective identification). Although for more precisely multiobjective identification the different functionals should be used, the algorithm proposed seems to be quite effective.

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