SOME REMARKS ON THE INFLUENCE OF INITIAL TEMPERATURE FLUCTUATIONS ON THE HEAT TRANSFER PROCESS IN MICROPERIODIC LAMINATED RIGID CONDUCTORS

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Abstract. In the note a certain aspects of the interrelation between initial temperature fluctuations and the averaged temperature field in the microperiodic laminated rigid conductors are investigated. The tolerance averaging technique is t into account as a tool of modelling. It was shown that there exist initial values of initial temperature fluctuations that the heat transfer process is independent of them.

1. Formulation of the problem

In the mathematical investigations of the averaged models of the heat transfer process in microperiodic rigid conductors the temperature field is represented in the form of

$$\theta = \vartheta + \theta_{res} \tag{1}$$

where ϑ is the averaged temperature field, [1, 2]. The crucial modelling problem is then strictly related to answer to the question how to introduce physical parameters or fields which with the sufficient approximation can determine the residual term $\theta_{res} \equiv \theta - \theta^0$. In the framework the well-known homogenization theory, [1], the residual term θ_{res} can be obtained by a solution to the unit cell problem and hence is uniquely determined by averaged temperature field. In the framework of the tolerance averaging approach, [2], the residual term θ_{res} is investigated in the form of a finite number of first terms of a certain Fourier expansion with respect to given *a priori* orthogonal function system h^A which are referred to as shape functions. Coefficients ϑ^A of this expansion, named *fluctuation amplitudes*, together with the averaged temperature field are new basic unknowns. Fluctuation amplitudes ϑ^A, $A = 1,...,N$, should be restricted by certain additional relations named physical reliability conditions [2]. The number N of the first terms of Fourier expansion which are taken into account and form of shape functions is strictly related to every problem which is analyzed by thew tolerance averaging technique application. In the framework of tolerance model the heat transfer in the microperiodic rigid conductor is represented by a system of partial differential equations of the first order with respect to time for fluctuation amplitudes and the averaged temperature. Hence, the problem of influence of initial temperature fluctuations on the heat transfer process in microperiodic laminated rigid conductors will be treated in this note as equivalent to the answer to the question what is the influence of initial and boundary temperature fluctuations onto a aforementioned heat transfer process.

The aim of this note is to discuss some special aspects of this problem in the framework of *simplified tolerance model*, [4], called also *second order tolerance model* [4] in which the evolution of fluctuation amplitudes is described by the separated system of differential equations of the first order with respect to time not depending on the averaged temperature field. In the framework of asymptotic homogenization model residual temperature field θ_{res} is uniquely determined by the averaged temperature field ϑ and hence the above problem cannot be correctly formulated.

2. Model equations

To make this paper self-consistent we are to shortly recall basic concepts of from the tolerance averaging technique (*TAT*). For details the reader is referred to [1]. The object of considerations to l-periodic conductors occupied in the refeence configuration the region $\Omega = \Pi \times (-L, L)$ in R^{n+1}, $n = 1, 2$, where Π is a certain regular region in R^n (in the special case of $n = 1$ region Π coincides with a certain real interval $(0, H)$, $H > 0$, cf. Figure 1. In the framework of the tolerance averaged model of heat conduction process in microperiodic rigid laminated conductors the residual temperature $\theta_{res}(\mathbf{x}, z, t)$ (at time instant *t* and in every point $(\mathbf{x}, z) \in \Pi \times (-L, L)$) is assumed in the form $\theta_{res}(\mathbf{x}, z, t) = g^A(z)\zeta^A(\mathbf{x}, z, t)$, $A = 1, ..., N$, where $g^1(\cdot)$, $g^2(\cdot)$, ... , $g^N(\cdot) \in O(l)$ are l-periodic piecewise differeniable continuous real shape functions postulated *a priori* in every problem formulated in the framework of a certain tolerance averaged model. Hence the total temperature is investigated in the form

$$\theta(\mathbf{x}, z, t) = \theta^0(\mathbf{x}, z, t) + g^1(z)\vartheta^1(\mathbf{x}, z, t) + g^2(z)\vartheta^2(\mathbf{x}, z, t) + ... + g^N(z)\vartheta^N(\mathbf{x}, z, t) \quad (2)$$

in which *averaged temperature* θ^0 and *fluctuation amplitudes* $\vartheta^1(\cdot, t)$, $\vartheta^2(\cdot, t)$, ... , $\vartheta^N(\cdot, t)$, should be slowly varying (together with all its derivatives applied in the model) functions in the 0z-axis direction. The thermal properties of the considered conductor will be described by the heat flux c and the conductive matrix $\mathbf{K}$ which are assumed to be l-periodic functions with respect to $z \in (-L, L)$ and constant in directions perpendicular to 0z-axis, i.e. $c(\mathbf{x}, z) = c(z)$, $\mathbf{K}(\mathbf{x}, z) = \mathbf{K}(z)$ for $(\mathbf{x}, z) \in \Pi \times (-L, L)$ and for every $(\mathbf{x}, z), (\mathbf{x}, z + l) \in \Pi \times (-L, L)$ we have $c(z) = c(z + l)$, $\mathbf{K}(z) = \mathbf{K}(z + l)$. Moreover, considerations will be restricted to the case in which conductive matrix $\mathbf{K}$ has the form

$$\mathbf{K} = \begin{bmatrix} \bar{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & k \end{bmatrix} \tag{3}$$

for a certain positive definite $n \times n$ symmetric matrix $\bar{\mathbf{K}} = \bar{\mathbf{K}}(z)$ and positive constant $k(z)$, which are l-periodic functions with respect to $z \in (-L, L)$. The starting point of the considerations is a system of tolerance averaged model equations of the heat transfer process, [1], which for the aforementioned laminated conductors will be written in the form

$$\begin{aligned} &\langle c \rangle \theta^0,_t - \bar{\nabla} \cdot \langle \bar{\mathbf{K}} \rangle \cdot \bar{\nabla} \theta^0 - \langle k \rangle \theta^0,_{zz} = [k]^A \vartheta^A,_z \\ &l^2 [(c)^{AB} \vartheta^B,_t - \bar{\nabla} \cdot (\bar{\mathbf{K}})^{AB} \cdot \bar{\nabla} \vartheta^A] + \{k\}^{AB} \vartheta^B = -[k]^A \theta^0,_z \end{aligned} \tag{4}$$

where we have introduced denotations:

$$\begin{aligned} &l^2 (c)^{AB} \equiv \langle c g^A g^B \rangle, \quad l^2 (\bar{\mathbf{K}})^{AB} \equiv \langle \bar{\mathbf{K}} g^A g^B \rangle \\ &[k]^A \equiv \langle k g^A,_z \rangle, \qquad \{k\}^{AB} \equiv \langle k g^A,_z g^B,_z \rangle \end{aligned} \tag{5}$$

The averaged temperature and fluctuation amplitudes being basic model unknowns should satisfy *physical reliability* conditions, cf. [1]

$$\theta^0(\mathbf{x}, \cdot, t), \ \vartheta^A(\mathbf{x}, \cdot, t) \in SV_l(T) \tag{6}$$

which means that $\theta^0(\mathbf{x}, \cdot, t)$ and $\vartheta^A(\mathbf{x}, \cdot, t)$ are slowly varying (together with all derivatives of these functions in the 0z-axis direction applied in (4)) with respect to the certain tolerance system T.

Due to the fact that $(4)_2$ treated as equations for ζ are time ordinary differential equations and partial differential equations with respect to the variable $\mathbf{x} \in \Pi$ we shall to formulate initial and boundary conditions for both basic unknowns ϑ and ζ. Assume the temperature initial condition in the form

$$\theta(\mathbf{x}, z, 0) = \bar{\theta}(\mathbf{x}, z), \ \mathbf{x} \in \Pi, \ z \in (-L, L) \tag{7}$$

where

$$\bar{\theta}(\mathbf{x},z)=\bar{\theta}^0(\mathbf{x},z)+g^A(\mathbf{x},z)\bar{\vartheta}^A(\mathbf{x},z),\ \mathbf{x}\in\Pi,\ z\in(-L,L) \tag{8}$$

for known slowly-varying functions $\bar{\theta}^0(\cdot)$ and $\bar{\vartheta}^A(\cdot)$ defined in Ω. Conditions (7) and (8) yields initial conditions for tolerance model equations (4)

$$\theta^0(\mathbf{x},z,0)=\bar{\theta}^0(\mathbf{x},z),\quad \vartheta^A(\mathbf{x},z,0)=\bar{\vartheta}^A(\mathbf{x},z) \tag{9}$$

Moreover, assume boundary conditions for a temperature field θ in the form

$$\theta(\mathbf{x},z,t)=\theta_b(\mathbf{x},z,t),\ (\mathbf{x},z)\in\partial\Omega,\ t\geq 0 \tag{10}$$

where

$$\theta_b(\mathbf{x},z,t)=\theta^0{}_b(\mathbf{x},z,t)+g^A(\mathbf{x})\vartheta^A{}_b(\mathbf{x},z,t),\ (\mathbf{x},z)\in\partial\Omega,\ t\geq 0 \tag{11}$$

for known slowly-varying functions $\theta^0{}_b(\cdot,t)$ and $\vartheta^A{}_b(\cdot,t)$ defined in $\partial\Omega$ for every time *t*. Conditions (10) and (11) yields boundary conditions for tolerance model equations (4)

$$\theta^0{}_b(\mathbf{x},z,t)=\bar{\theta}^0(\mathbf{x},z,t),\quad \vartheta^A{}_b(\mathbf{x},z,t)=\bar{\vartheta}^A(\mathbf{x},z,t),\ \mathbf{x}\in\Omega,\ z\in(-L,L),\ t\geq 0 \tag{12}$$

We shall also assume that

$$\theta^0{}_b(\mathbf{x},z,0)=\bar{\theta}^0(\mathbf{x},z),\quad \vartheta^A{}_b(\mathbf{x},z,0)=\bar{\vartheta}^A(\mathbf{x},z),\ \mathbf{x}\in\partial\Omega \tag{13}$$

Tolerance model equations (4), initial/boundary conditions (9), (12) together with (13) will be referred to as *the tolerance averaged model for heat conduction process in l-periodic conductor under consideration*.
To simplify considerations we shall assume that conductivity matrix has the form

$$\mathbf{K}(z)=\mathbf{\Psi}\,c(z),\quad z\in(-L,L) \tag{14}$$

where

$$\mathbf{\Psi}=\begin{bmatrix}\bar{\mathbf{\Psi}} & \mathbf{0}\\ \mathbf{0} & \psi\end{bmatrix} \tag{15}$$

for a certain positive $n\times n$ symmetric matrix $\bar{\mathbf{\Psi}}$ and positive reals ψ. Condition (14) will be referred to as *the heat proportional assumption.* Hence, we have

$$(\bar{\mathbf{K}})^{AB}\equiv\bar{\mathbf{\Psi}}(c)^{AB},\qquad [k]^A\equiv\psi[c]^A,\qquad \{k\}^{AB}\equiv\psi\{c\}^{AB} \tag{16}$$

for

$$[c]^A = \langle cg^A{,}_z \rangle, \qquad \{c\}^{AB} = \langle cg^A{,}_z\, g^B{,}_z \rangle \tag{17}$$

Under denotation

$$\overline{\nabla}_\psi \equiv \overline{\nabla} \cdot \overline{\mathbf{\Psi}} \cdot \overline{\nabla} \tag{18}$$

model equations (4) yields

$$\begin{aligned} &\langle c\rangle(\theta^0{,}_t - \overline{\nabla}_\psi \theta^0 - \psi\theta^0{,}_{zz}) = \psi[c]^A \vartheta^A{,}_z \\ &l^2(c)^{AB}(\vartheta^B{,}_t - \overline{\nabla}_\psi \vartheta^A) + \psi\{c\}^{AB}\vartheta^B = -\psi[c]^A\theta^0{,}_z \end{aligned} \tag{19}$$

In the subsequent considerations constant matrix (15) will be treated as a certain parameter characterized thermal properties of the considered conductor.

3. Analysis

Now we are to show that, under certain assumptions, fluctuation amplitudes $\zeta(\cdot)$ can be described by the system of equations which is independent of the averaged temperature ϑ. Considerations can be treated as special case of those for the hyperbolic heat transfer process in the periodic rigid conductors which yield the simplified tolerance averaged model and is explained in [3]. To formulate aforementioned assumptions we shall introduce *gradient part* ζ_{hom} *of the fluctuation amplitude* of fluctuation amplitude ζ

$$\vartheta^A_{hom} = -(\{c\}^{-1})^{AB}[c]^B \vartheta{,}_z \tag{20}$$

where

$$(\{c\}^{-1})^{AB}\{c\}^{BC} = \delta^{AC} \tag{21}$$

for $A, B, C = 1,...,N$. Hence we have decomposed every fluctuation amplitude $\overline{\vartheta}^A$ onto two terms

$$\vartheta^A = \vartheta^A_{hom} + \chi^A \tag{22}$$

From the second of model equations (19) terms χ^A of the above decomposition, which will be referred to as *fluctuation variables*, should satisfy partial differential equations

$$l^2(c)^{AB}[\chi^B{,}_t - \overline{\nabla}_\psi \chi^B] + \psi\{c\}^{AB}\chi^B = l^2(c)^{AB}(\overline{\nabla}_\psi \vartheta^B_{hom} - \vartheta^B_{hom}{,}_t) - \psi\{c\}^{AB}\vartheta^B_{hom}{,}_z \tag{23}$$

At the same time the first from equations (4) takes the form

$$\langle c\rangle(\vartheta,_t - \overline{\nabla}_\psi \vartheta) - k^{eff}\vartheta,_{zz} = \psi[c]^A \chi^A,_z \tag{24}$$

where

$$k^{eff} = \psi c^{eff} \tag{25}$$

for

$$c^{eff} = \langle c\rangle - [c]^A H^{BC}[c]^A \tag{26}$$

It can be shown, cf. [3], that if all derivatives of fluctuation amplitudes $\overline{\vartheta}^A$ applied in (4) (not only in the 0*z*-axis direction) are slowly varying then the right hand side of (23) can be omitted and hence the second term χ^A in (22) should satisfy ordinary differential equation

$$l^2(c)^{AB}(\chi^B,_t - \overline{\nabla}_\psi \chi^B) + \psi\{c\}^{AB}\chi^B = 0 \tag{27}$$

and hence boundary conditions for χ^A, written in (12), cannot be taken into account. Under (20) and (13) initial conditions for χ^A takes the form

$$\chi^A(\mathbf{x}, z, 0) = \overline{\chi}^A(\mathbf{x}, z),\ \mathbf{x} \in \Omega,\ \ z \in (-L, L) \tag{28}$$

where under (20) $\chi_o{}^A$ is interrelated with initial value ϑ_0 of averaged temperature ϑ by

$$\overline{\chi}^A(\mathbf{x}, z) = \overline{\vartheta}^A(\mathbf{x}, z) + \psi H^{AB}[c]^B \nabla\theta^0(\mathbf{x}, z), \quad \mathbf{x} \in \Pi,\ \ z \in (-L, L) \tag{29}$$

and will be referred to as *the initial fluctuation*. Let us observe that for given initial condition $(9)_2$ fluctuation variable χ can be determined from (27). Indeed, introducing for every square matrix $\mathbf{A}$ denotation

$$\exp(\mathbf{A}) \equiv \sum_{n=0}^{+\infty} \frac{\mathbf{A}^n}{n!} \tag{30}$$

we conclude that model equations (19) can be reduced to the form single heat conduction equation for averaged temperature

$$\langle c\rangle(\theta^0,_t - \overline{\nabla}_\psi \theta^0) - k^{eff}\theta^0,_{zz} = s(\mathbf{x}, t) \tag{31}$$

where $\mathbf{H}_c$ is a $N \times N$ positive matrix defined by

$$H_c{}^{AC} \equiv \psi((c)^{-1})^{AB}\{c\}^{BC}, \quad A, B, C = 1, ..., N \tag{32}$$

and

$$s(\mathbf{x},z,t)=\psi[c]^A \exp(-l^2\mathbf{H}_c t)^{AB}\,\overline{\chi}^B{,}_z(\mathbf{x},z) \tag{33}$$

Equation (31) will be referred to as *the mean heat conduction equation* while function (33) will be called *a temperature pseudosource*. Formula (2) in the considered case takes the form

$$\theta(\mathbf{x},z,t)=\theta^0(\mathbf{x},z,t)+g^A(z)\left[H^{AB}[c]^B\theta^0{,}_z(\mathbf{x},z,t)+\exp(-l^2\mathbf{H}_c t)^{AB}\,\overline{\chi}^B(\mathbf{x},z)\right] \tag{34}$$

It is easy to verify that the pseudosource (33) is a value of differential form

$$\psi[c]^A \exp(-l^2\mathbf{H}_c t)^{AB}\partial_z \tag{35}$$

on initial values $\overline{\chi}^A=\overline{\chi}^A(\mathbf{x},z)$. Denote

$$S_t^{\perp}=\{\overline{\chi}^B=(\mathbf{x},z):\psi[c]^A \exp(-l^2\mathbf{H}_c t)^{AB}\,\overline{\chi}^B{,}_z(\mathbf{x},z)=0\} \tag{36}$$

Every initial values $\overline{\chi}^A=\overline{\chi}^A(\mathbf{x},z)$ which do not depend on the variable z belongs $S_t^{\perp}$. Set $S_t^{\perp}$ is a certain linear space. It easy to verify that $S_t^{\perp}$ includes also initial values $\overline{\chi}^A=\overline{\chi}^A(\mathbf{x},z)$ which satisfy condition

$$\begin{aligned}\overline{\chi}^N(\mathbf{x},z)=-\kappa(z)[c]^A\{&\exp(-l^2\mathbf{H}_c t)^{A1}\,\overline{\chi}^1(\mathbf{x},z)+\ldots\\&\ldots+\exp(-l^2\mathbf{H}_c t)^{AN-1}\,\overline{\chi}^{N-1}(\mathbf{x},z)\}\end{aligned} \tag{37}$$

for

$$\kappa(z)=\frac{\overline{\kappa}(z)}{[c]^A \exp(-l^2\mathbf{H}_c t)^{AN}}$$

and for an arbitrary differentiable function $\overline{\kappa}=\overline{\kappa}(z)$. Since for every element from $S_t^{\perp}$ the right hand side the mean heat conduction equation (31) vanishes the differential form (35) plays role of a certain filter for initial values $\overline{\chi}^A=\overline{\chi}^A(\mathbf{x},z)$. Every instant t for which $\overline{\chi}^A(\mathbf{x},z)\in S_t^{\perp}$ will be called *an instant of the filtering*. Moreover, the right hand side of the mean heat conduction equation (31) can be treated as a value of a certain bilinear form in $\psi[c]^A$ and $\partial_z\overline{\chi}^B$. This form is positive and symmetric since their matrix $\exp(-l^2\mathbf{H}_c t)$ can be represented as

$$\exp(-l^2\mathbf{H}_c t)=\mathbf{O}^T diag(e^{-l^2\lambda^{(1)}t},\ldots,e^{-l^2\lambda^{(N)}t})\mathbf{O} \tag{38}$$

where $\lambda^{(1)},...,\lambda^{(N)}$ are eigenvalues of positive and symmetric matrix $\mathbf{H}_c$ and $\mathbf{O}$ is certain orthonormal matrix. Hence, if l tends to zero then pseudosource also tends to zero.

4. Final remarks

Summing up aforementioned considerations it should be emphasized that in the framework of the simplified tolerance averaged model of the heat transfer in microperiodic laminated rigid conductors can be reformulated to the form of classical heat transfer equation with constant coefficients which depends on the fluctuations of initial temperature by the source term named in the paper as pseudosource. The pseudosource has the following properties:

1° it depends on the derivative in the $0z$-axis direction of fluctuation amplitude initial values (28),

2° it monotonically degrees in the time being,

3° it is equal to zero provided that the considered conductor is a homogeneous one or initial fluctuation function $\overline{\chi}^A(\cdot)$ do not depend on z or has a special form given by (37),

4° it is a certain filter for initial values of fluctuation amplitudes, i.e. there exist initial values $\overline{\chi}^A(\cdot)$, $A=1,...,N$, of fluctuation amplitudes on which the heat transfer in microperiodic laminated rigid conductors cannot depends (in the framework of simplified tolerance model).

It can be emphasized that there exist families of initial values of fluctuation amplitudes on which the heat transfer in microperiodic laminated rigid conductors depends identically. This families are spaces of the form $[\overline{\chi}^1(\cdot),...,\overline{\chi}^N(\cdot)]+S_t^{\perp}$ for a given sufficiently regular initial values $[\overline{\chi}^1(\cdot),...,\overline{\chi}^N(\cdot)]$.

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