IDENTIFICATION OF TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY USING THE GRADIENT METHOD

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Abstract. In the paper the application of the BEM for numerical solution of the inverse parametric problem is presented. On the basis of the knowledge of temperature field in the domain considered the temperature dependent thermal conductivity is identified. The steady state is considered (2D problem) and mixed boundary conditions are taken into account. The inverse problem is solved using the gradient method. In the final part of the paper the results of computations are shown.

1. Formulation of the problem

The following 2D problem is considered

$$\begin{aligned} x\in\Omega: &\quad \nabla[\lambda(T)\nabla T(x)]=0 \\ x\in\Gamma_1: &\quad T(x)=T_b(x) \\ x\in\Gamma_2: &\quad q(x)=-\lambda(T)\,\mathbf{n}\cdot\nabla T(x)=q_b(x) \end{aligned} \tag{1}$$

where T is the temperature, $x = (x_1,x_2)$ are the spatial coordinates, $\lambda(T)$ is the thermal conductivity, $T_b(x)$, $q_b(x)$ are known boundary temperature and boundary heat flux, **n** is the normal outward vector at the boundary point x. We assume that

$$\lambda(T)=c_1T+c_2 \tag{2}$$

where c_1, c_2 are the coefficients.

If the direct problem is considered then all geometrical and thermophysical parameters appearing in the mathematical model (1) are known.

In the paper the inverse parametric problem is considered in which we assume that the coefficients c_1, c_2 are unknown. In order to solve the inverse problem the additional information is necessary. So, we assume that the temperatures at the selected points $x_i\in\Omega$ are given

$$T_{d\,i}=T_d(x_1^i,x_2^i), \qquad i=1,2,\dots,M \tag{3}$$

where M is the number of sensors.

2. Solution of direct problem using the boundary element method

In order to solve the problem (1), (2) for arbitrary assumed values of parameters c_1, c_2 the Kirchhoff transformation is introduced

$$U(T) = \int_0^T \lambda(\mu)\, \mathrm{d}\mu \tag{4}$$

and then the governing equations (1) take a form

$$\begin{cases} x \in \Omega: & \nabla^2 U(x) = 0 \\ x \in \Gamma_1: & U(x) = U_b(x) = U(T_b) \\ x \in \Gamma_2: & q(x) = -\mathrm{n} \cdot \nabla U(x) = q_b(x) \end{cases} \tag{5}$$

where (c.f. equation (2))

$$U(x) = \frac{1}{2} c_1 T^2(x) + c_2\, T(x) \tag{6}$$

The integral equation corresponding to the problem (5) is the following [1, 2]

$$B(\xi)\, U(\xi) + \int_\Gamma U^*(\xi, x)\, q(x)\, \mathrm{d}\Gamma = \int_\Gamma Q^*(\xi, x)\, U(x)\, \mathrm{d}\Gamma \tag{7}$$

where ξ is the observation point, $B(\xi)$ is the coefficient from the scope (0, 1], $U^*(\xi,x)$ is the fundamental solution and for 2D domain oriented in Cartesian coordinate system it is a function of the form

$$U^*(\xi, x) = \frac{1}{2\pi} \ln \frac{1}{r} \tag{8}$$

where r is the distance between the points ξ and x.
The function

$$Q^*(\xi, x) = -\mathrm{n} \cdot \nabla U^*(\xi, x) \tag{9}$$

is calculated in analytic way and then

$$Q^*(\xi, x) = \frac{d}{2\pi \mathrm{r}^2} \tag{10}$$

where

$$d = (x_1 - \xi_1)\cos\alpha_1 + (x_2 - \xi_2)\cos\alpha_2 \tag{11}$$

and $\cos\alpha_1$, $\cos\alpha_2$ are the directional cosines of the normal outward vector **n**.

In numerical realization of the BEM the boundary Γ is divided into N constant boundary elements Γ_j, $j = 1,...,N$ and then the approximation of equation (7) takes a form

$$B_i U_i + \sum_{j=1}^{N} q_j \int_{\Gamma_j} U^*(\xi^{\mathrm{i}}, x)\mathrm{d}\Gamma_{\mathrm{j}} = \sum_{j=1}^{N} U_j \int_{\Gamma_j} Q^*(\xi^{\mathrm{i}}, x)\mathrm{d}\Gamma_{\mathrm{j}} \tag{12}$$

or

$$B_i U_i + \sum_{j=1}^{N} G_{ij} q_j = \sum_{j=1}^{N} H_{ij} U_j \tag{13}$$

where

$$G_{ij} = \int_{\Gamma_j} U^*(\xi^i, x)\, \mathrm{d}\Gamma_j \tag{14}$$

while

$$H_{ij} = \int_{\Gamma_j} Q^*(\xi^i, x)\, \mathrm{d}\Gamma_j \tag{15}$$

and $B_i = B(\xi^i)$, $U_i = U(\xi^i)$, $U_j = U(x^j)$, $q_j = q(x^j)$. It should be pointed out that for $\xi^i \in \Gamma_j$: $B_i = 1/2$ (Γ_j is the constant boundary element), while for $\xi^i \in \Omega$: $B_i = 1$.
For $i = 1,2,...,N$ one obtains the system of N equations of type (13) from which the 'missing' boundary values U_j and q_j can be determined.

The values of function U_i, $i = N+1$, $N+2,...,N+L$ at the internal points are calculated using the formula (c.f. equation (13))

$$U_i = \sum_{j=1}^{N} H_{ij} U_j - \sum_{j=1}^{N} G_{ij} q_j \tag{16}$$

The obtained internal values of function U should be re-counted:

$$\frac{1}{2} c_1 T_i^2 + c_2 T_i - U_i(T_i) = 0 \tag{17}$$

One of the roots of this equation corresponds to the searched value of temperature T_i.

3. Sensitivity analysis with respect to c_1 and c_2

Taking into account the dependence (6) the integral equation (7) can be written as follows

$$
\begin{aligned}
&B(\xi)\left[\frac{1}{2}c_1T^2(\xi)+c_2\,T(\xi)\right]+\int_\Gamma U^*(\xi,x)\,q(x)\,\mathrm{d}\Gamma\ =\\
&\int_\Gamma Q^*(\xi,x)\ \left[\frac{1}{2}c_1T^2(x)+c_2\,T(x)\right]\ \mathrm{d}\Gamma
\end{aligned}
\tag{18}
$$

We differentiate the equation (18) with respect to c_1 and then

$$
\begin{aligned}
&B(\xi)\left[\frac{1}{2}T^2(\xi)+c_1\,T(\xi)Z_1(\xi)+c_2Z_1(\xi)\right]+\int_\Gamma U^*(\xi,x)\,Q_1(x)\,\mathrm{d}\Gamma\ =\\
&\int_\Gamma Q^*(\xi,x)\ \left[\frac{1}{2}T^2(x)+c_1\,T(x)Z_1(x)+c_2\,Z_1(x)\right]\ \mathrm{d}\Gamma
\end{aligned}
\tag{19}
$$

where $Z_1(x)=\partial T(x)/\partial c_1$, $Q_1(x)=\partial q(x)/\partial c_1$.
The approximate form of equation (19) is following

$$
B_i\ \left(\frac{1}{2}T_i^2+\lambda_iZ_{i1}\right)+\sum_{j=1}^{N}\ G_{i\,j}Q_{j1}=\sum_{j=1}^{N}\ H_{i\,j}\left(\frac{1}{2}T_j^2+\lambda_jZ_{j1}\right)
\tag{20}
$$

or

$$
\sum_{j=1}^{N}\ G_{ij}Q_{j1}=\sum_{j=1}^{N}\ H_{ij}\lambda_jZ_{j1}-B_i\lambda_iZ_{i1}+\sum_{j=1}^{N}\frac{1}{2}H_{ij}T_j^2-\frac{1}{2}B_iT_i^2
\tag{21}
$$

Next, the equation (18) is differentiated with respect to c_2

$$
\begin{aligned}
&B(\xi)\,[c_1T(\xi)Z_2(\xi)+T(\xi)+c_2\mathrm{Z}_2(\xi)]+\int_\Gamma U^*(\xi,x)Q_2(x)\mathrm{d}\Gamma=\\
&\int_\Gamma Q^*(\xi,x)[c_1T(x)Z_2(x)+T(x)+c_2\mathrm{Z}_2(x)]\mathrm{d}\Gamma
\end{aligned}
\tag{22}
$$

where $Z_2(x)=\partial T(x)/\partial c_2$ and $Q_2(x)=\partial q(x)/\partial c_2$.
The approximate form of equation (22) is following

$$
B_i(T_i+\lambda_iZ_{i2})+\sum_{j=1}^{N}\ G_{ij}Q_{j2}=\sum_{j=1}^{N}\ H_{ij}(T_j\ +\lambda_jZ_{j2})
\tag{23}
$$

or

$$
\sum_{j=1}^{N}\ G_{ij}\ Q_{j2}=\sum_{j=1}^{N}\ H_{ij}\ \lambda_jZ_{j2}-B_i\lambda_iZ_{i2}+\sum_{j=1}^{N}\ H_{ij}\,T_j-B_iT_i
\tag{24}
$$

The boundary conditions are also differentiated with respect to c_e, e = 1,2, and then

$$
\begin{aligned}
x \in \Gamma_1: &\qquad Z_e(x) = 0 \\
x \in \Gamma_2: &\qquad Q_e(x) = \frac{\partial\, q(x)}{\partial\, c_e} = 0
\end{aligned}
\tag{25}
$$

So, in order to determine the sensitivity functions Z_{j1} and Z_{j2} at the boundary nodes x^j, j = 1,2,...,N two systems of equations (21), (24) should be solved. Next, the values of functions Z_{i1} and Z_{i2} at the internal points x^i are calculated using again the formulas (21), (24) for $i = N+1, N+2,...,N+L$. It should be pointed out that the additional problems (21, (24) are coupled with the basic problem (13) because the values of temperatures T_j appearing in (21), (24) should be known.

4. Solution of inverse problem

In order to solve the inverse problem the following least squares criterion is applied [3, 4]

$$
S\,(c_1, c_2) = \sum_{i=1}^{M} \left(T_i - T_{di}\right)^2
\tag{26}
$$

where $T_i = T\left(x_1^i, x_2^i\right)$ is the calculated temperature T at the point x_i for arbitrary assumed values of c_1 and c_2.

The necessary condition of optimum of function (26) leads to the following system of equations

$$
\begin{cases}
\dfrac{\partial\, S}{\partial\, c_1} = 2\displaystyle\sum_{i=1}^{M} (T_i - T_{d\,i})\,\dfrac{\partial\, T_i}{\partial\, c_1}\ \Big|_{c_1 = c_i^k} = 0 \\[2ex]
\dfrac{\partial\, S}{\partial\, c_2} = 2\displaystyle\sum_{i=1}^{M} (T_i - T_{d\,i})\,\dfrac{\partial\, T_i}{\partial\, c_2}\ \Big|_{c_2 = c_2^k} = 0
\end{cases}
\tag{27}
$$

where c_1^k, c_2^k for $k = 0$ are the arbitrary assumed values of parameters c_1, c_2, while c_1^k, c_2^k for $k > 0$ result from the previous iteration.

Function T_i is expanded into the Taylor series about known values of c_1^k, c_2^k

$$
T_i = T_i^k + \frac{\partial\, T_i}{\partial\, c_1}\ \Big|_{c_1 = c_1^k} \left(c_1^{k+1} - c_1^k\right) + \frac{\partial\, T_i}{\partial\, c_2}\ \Big|_{c_2 = c_2^k} \left(c_2^{k+1} - c_2^k\right)
\tag{28}
$$

this means

$$T_i = T_i^k + Z_{i1}^k \left(c_1^{k+1} - c_1^k\right) + Z_{i2}^k \left(c_2^{k+1} - c_2^k\right) \tag{29}$$

where

$$Z_{i1}^k = \frac{\partial T_i}{\partial c_1} \Big|_{c_1 = c_1^k}, \qquad Z_{i2}^k = \frac{\partial T_i}{\partial c_2} \Big|_{c_2 = c_2^k} \tag{30}$$

are the sensitivity coefficients. Putting (29) into (27) one obtains

$$\begin{cases} \sum\limits_{i=1}^{M} \left[T_i^k + Z_{i1}^k \left(c_1^{k+1} - c_1^k\right) + Z_{i2}^k \left(c_2^{k+1} - c_2^k\right) - T_{d\,i}\right] Z_{i1}^k = 0 \\ \sum\limits_{j=1}^{M} \left[T_i^k + Z_{i1}^k \left(c_1^{k+1} - c_1^k\right) + Z_{i2}^k \left(c_2^{k+1} - c_2^k\right) - T_{d\,i}\right] Z_{i2}^k = 0 \end{cases} \tag{31}$$

or

$$\begin{cases} \sum\limits_{i=1}^{M} \left[Z_{i1}^k \left(c_1^{k+1} - c_1^k\right) + Z_{i2}^k \left(c_2^{k+1} - c_2^k\right)\right] Z_{i1}^k = \sum\limits_{i=1}^{M} \left(T_{d\,i} - T_i^k\right) Z_{i1}^k \\ \sum\limits_{j=1}^{M} \left[Z_{i1}^k \left(c_1^{k+1} - c_1^k\right) + Z_{i2}^k \left(c_2^{k+1} - c_2^k\right)\right] Z_{i2}^k = \sum\limits_{i=1}^{M} \left(T_{d\,i} - T_i^k\right) Z_{i2}^k \end{cases} \tag{32}$$

where $k = 1,...,K$ is the number of iteration.
This system of equations can be written in the matrix form

$$(\mathrm{Z}^{\mathbf{T}})^{\boldsymbol{k}}\, \mathrm{Z}^{\boldsymbol{k}} \mathrm{c}^{\boldsymbol{k+1}} = (\mathrm{Z}^{\mathbf{T}})^{\boldsymbol{k}}\, \mathrm{Z}^{\boldsymbol{k}} \mathrm{c}^{\boldsymbol{k}} + (\mathrm{Z}^{\mathbf{T}})^{\boldsymbol{k}}\, (\mathrm{T}_{\boldsymbol{d}} - \mathrm{T}^{\boldsymbol{k}}) \tag{33}$$

where

$$\mathbf{Z}^k = \begin{bmatrix} Z_{11}^k & Z_{12}^k \\ Z_{21}^k & Z_{22}^k \\ \cdots & \cdots \\ Z_{M1}^k & Z_{M2}^k \end{bmatrix} \tag{34}$$

and

$$\mathbf{T}_d = \begin{bmatrix} T_{d\,1} \\ T_{d\,2} \\ \cdots \\ T_{d\,M} \end{bmatrix}, \qquad \mathbf{T}^k = \begin{bmatrix} T_1^k \\ T_2^k \\ \cdots \\ T_M^k \end{bmatrix} \tag{35}$$

while

$$\mathbf{c}^k = \begin{bmatrix} c_1^k \\ c_2^k \end{bmatrix}, \quad \mathbf{c}^{k+1} = \begin{bmatrix} c_1^{k+1} \\ c_2^{k+1} \end{bmatrix} \tag{36}$$

The system of equations (33) allows to determine the values of c_1^{k+1}, c_2^{k+1}. The iteration process is stopped when the assumed number of iterations is achieved.

5. Example of computations

The square domain of dimensions 0.1x0.1 m has been considered. On the left surface the boundary heat flux $q_b = -5 \cdot 10^6$ W/m^2 has been assumed, on the remaining parts of the boundary the temperature 100^{o}C has been accepted. The boundary has been divided into 40 constant boundary elements.
At first the direct problem has been solved under the assumption that

$$\lambda(T) = -0.05427\,T + 386.116 \tag{37}$$

In Figure 1 the position of isotherms 120, 140,..., etc. is shown.
Next, the inverse problem has been considered. It is assumed that the temperatures at four internal nodes (0.02, 0.05), (0.04, 0.05), (0.06, 0.05), (0.08, 0.05) (Fig. 1) are given.

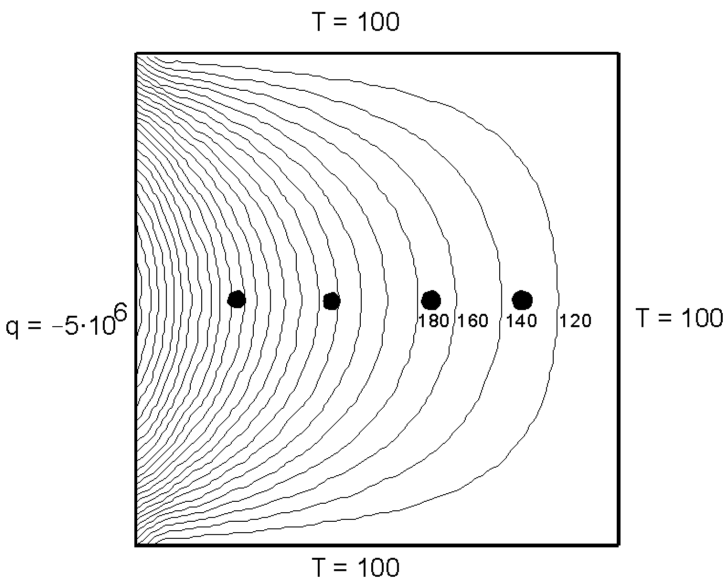

Fig. 1. Temperature distribution

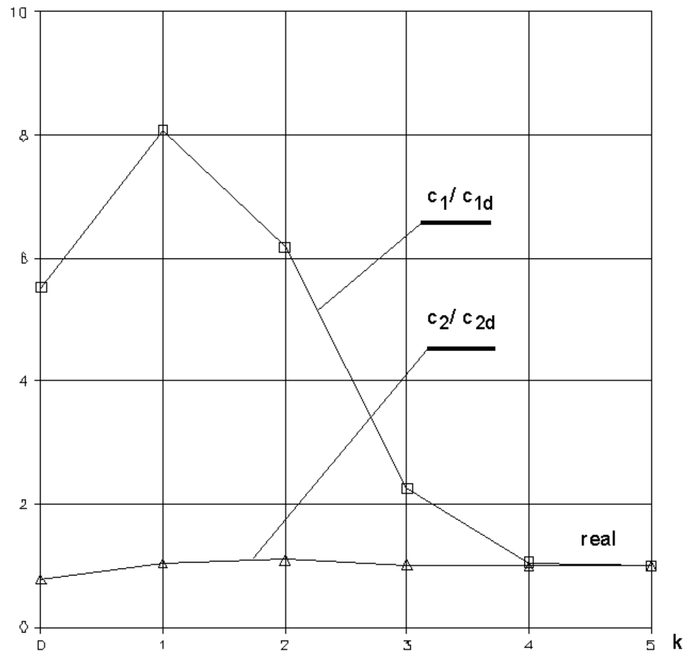

Fig. 2. Identification of parameters for $c_1^0 = -0.3, c_2^0 = 300$

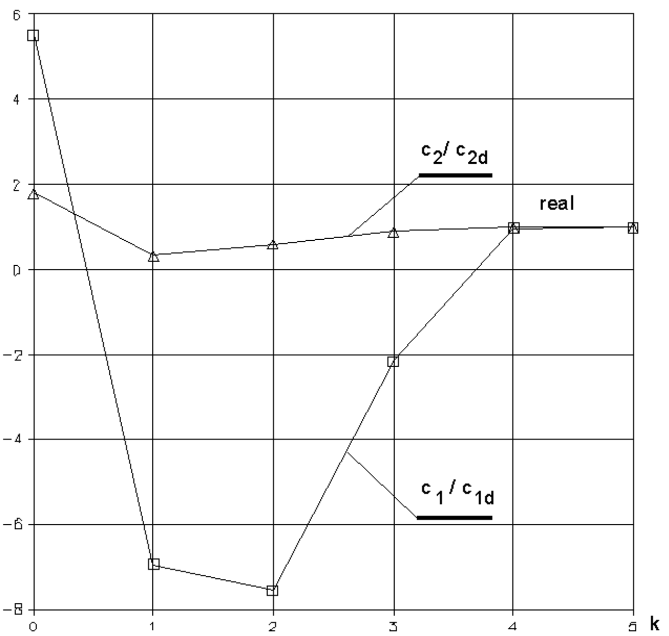

Fig. 3. Identification of parameters for $c_1^0 = -0.3, c_2^0 = 700$

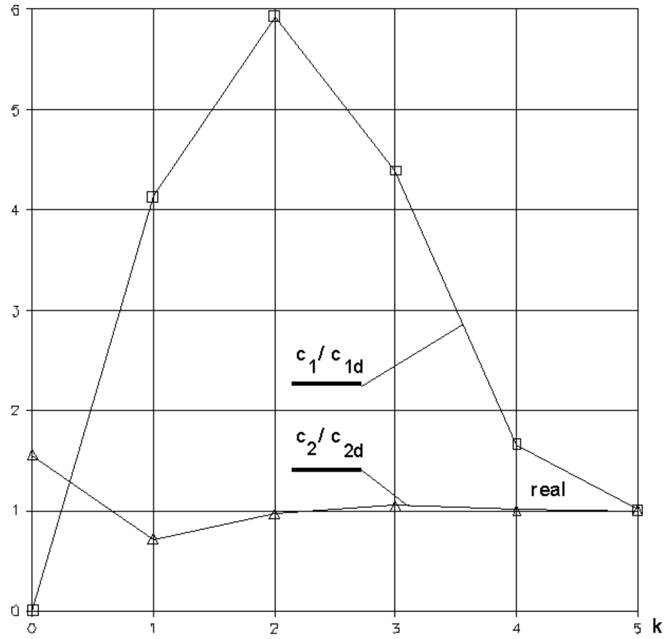

Fig. 4. Identification of parameters for $c_1^0 = 0, c_2^0 = 600$

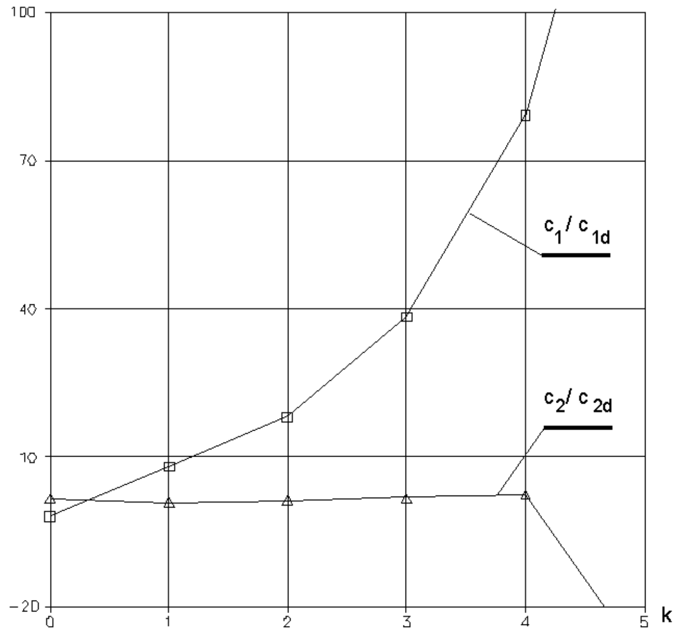

Fig. 5. Identification of parameters for $c_1^0 = -0.1, c_2^0 = 600$ - iteration process is not convergent

In Figures 2-4 the results of identification of parameters c_1/c_{1d}, c_2/c_{2d} (c_{1d} = –0.05426, c_{2d} = 386.116 denote the real values of these parameters) for different initial values c_1^0, c_2^0 are shown. It is visible that the iteration process is convergent and after 5 iterations the real values of searched parameters are obtained.

It should be pointed out that iteration process is not always convergent, for example for $c_1^0 = 0.1$ and $c_2^0 = 600$ - c.f. Figure 5. So, the proper choice of initial values of identified parameters allows the convergence of iteration procedure.
Summing up, the algorithm presented allows to identify the unknown parameter as the temperature dependent function, and it is the main advantage of the approach discussed.

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