TOROIDAL APPROXIMATION FOR CAPILLARY BRIDGES BETWEEN ELLIPSOIDAL GRAINS

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Abstract. The capillary bridge between ellipsoidal grains is modelled within a toroidal approximation and the cohesive energy as function of grain's size, wetting angle and liquid content is computed. Some substrate-liquid examples are considered.

Granular materials are common in Nature and they exhibits a variety of unusual phenomena [1]. It is known that many mechanical properties of a granulate change if some liquid is added. Even small amount of interstitial liquid adds an attractive force to the system and then, increase its stability [2-10]. A study of the effects of liquid on static and dynamic properties of dense two-dimensional grain ensembles is of interest in many branches of industry because humidity causes problem in processes such as segregation, transport or packing. The main reason is the internal cohesion due to capillary forces arising from liquid bridges between the grains.

To describe the behaviour of wetting liquid and solid state grain we assume that grains have porous surfaces. Capillary bridges need a certain amount of liquid content to form properly, since the liquid is at first bound on the grain surfaces due to the roughness. Two regimes of the inter-grain adhesive force versus volume of the wetting layer have been considered: the asperity regime for very small amount of liquid and the saturated regime for larger content of liquid. In case of asperity regime the capillary forces come from the fluid accumulated around small number of asperities at which two neighbouring grains in contact. This case has been already discussed in [11]. If all asperities are filled we deal with saturated regime.

Most of grains are not spherical. Thus, it is important to analyze how the macroscopic curvature of contacted grains influences on an inter-grain adhesive energy. Here we consider the ellipsoidal grains. For this shape of grain there are three different grain - to - grain arrangements presented in Figure 1.

The geometry of the liquid bridge is characterized by wetting angle, surface tension of liquid, size and shape of grains. Commonly used approach is based on the Laplace-Young equation

$$\frac{\Delta P}{\gamma} = \frac{\dfrac{d^2 y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} - \frac{1}{y\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{1}{2}}} \tag{1}$$

where ΔP is the capillary suction pressure, γ is the liquid surface tension and unknown function $y(x)$ is a curvature of the capillary bridge. The Equation (1) cannot be solved analytically [12].

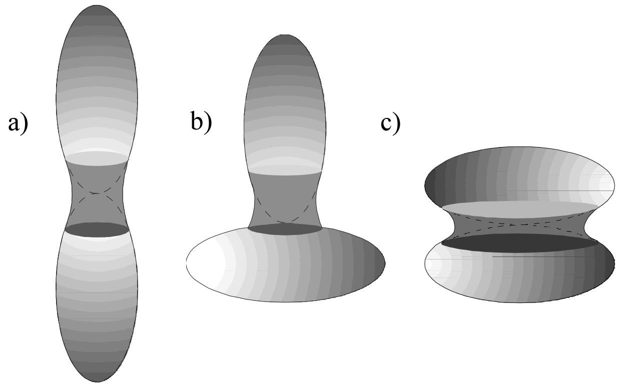

Fig. 1. Essentially different arrangements of ellipsoidal grains for saturated regime

In Figure 1, for the configuration (a) the rotational symmetry of grains and liquid is conserved and so we can approximate the complicated free - liquid surface by simpler surface, i.e. by a toroid. Within the toroidal approximation the Equation (1) becomes

$$\Delta P = \gamma\left(\frac{1}{r_1} - \frac{1}{r}\right) \tag{2}$$

Here r_1 is the radius of curvature of the liquid bridge at point P in the horizontal plane and r is the radius of the curvature in the vertical plane going through the axis of symmetry. An approximate, toroidal capillary bridge is presented in Figure 2.

The geometry of the problem is characterized by wetting angle θ, liquid volume V_k and two semi-axis: the minor axis α and the major axis $b = k\alpha$, $k \geq 1$. Cross-section of the grain and the capillary bridge gives an ellipse and a circle:

$$\frac{x^2}{a^2} + \frac{(y_1 - b)^2}{b^2} = 1 \tag{3}$$

$$(x-d)^2 + y_2^2 = r^2 \tag{4}$$

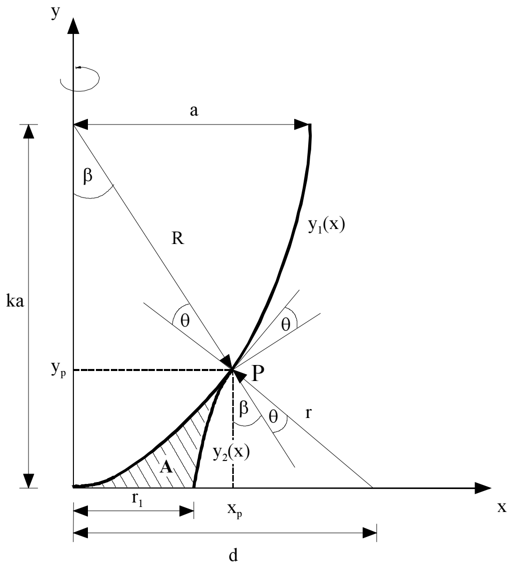

Fig. 2. The fragment of a cut of ellipsoidal grain and liquid bridge for the (a) arrangement. The shaded area *A* is filled with wetting liquid [13]

In order to determine a lateral surface of the liquid bridge it is necessary to estimate the length of an arc between points $P(x_p, y_p)$ located on two grains. The length of this arc depends on volume V_k of the capillary bridge. From Equations (3), (4) and Figure 2, we obtain:

$$x_p = R\sin\beta,\ \ y_p = ka - R\cos\beta \tag{5}$$

$$R = \frac{ka}{\sqrt{k^2\sin^2\beta + \cos^2\beta}} \tag{6}$$

$$x_p = d - r\sin(\beta+\theta),\ \ y_p = r\cos(\beta+\theta) \tag{7}$$

Comparing (5) and (7) we get the relation between the liquid content V_k (given by R, β, θ) and the grain size (given by α, k)

$$r = \frac{ka - R\cos\beta}{\cos(\beta+\theta)} \tag{8}$$

So, using the Guldin-Pappus theorem and the Equation (8) we compute the free surface of the liquid bridge S_b

$$S_b = 4\pi a^2 k\frac{I_1 - I_2}{A_1 - A_2}\left(\frac{\pi}{2} - \arctan(kc)\right)\left(1 - \frac{1}{\sqrt{1+c^2}}\right)\sqrt{1+(kc)^2} \tag{9}$$

where:

$$A_1 = \frac{c}{\sqrt{1+c^2}} - \frac{1}{2}\arctan(c) - \frac{c}{2(1+c^2)}$$

$$A_2 = \frac{k}{2}\left(1 - \frac{1}{\sqrt{1+c^2}}\right)^2 \left(1+(kc)^2\right)\left(\frac{\pi}{2} - \arctan(kc) - \frac{kc}{1+(kc)^2}\right)$$

$$I_1 = \frac{c^2}{2(1+c^2)} + \frac{1}{3(1+c^2)^{3/2}} - \frac{1}{3}$$

$$I_2 = \frac{k}{2}\left(1 - \frac{1}{\sqrt{1+c^2}}\right)^2 \left[c\left(k^2 - \frac{k^2-1}{\sqrt{1+c^2}}\right)\left(1+(kc)^2\right)\left(\frac{\pi}{2} - \arctan(kc) - \frac{kc}{1+(kc)^2}\right) - \frac{k}{3}\left(1 - \frac{1}{\sqrt{1+c^2}}\right)\right]$$

and $c = k\tan(\beta)$.
The cohesive energy E for the A-A grain arrangement (see Figure 1) is given by

$$E = \gamma S_b \tag{10}$$

where γ is the liquid surface tension and S_b is given by the Equation (9).

In Table 1 the values of S_b are given for some θ's grain size α, k for the liquid amount $V_k = 10^{-2}$ of the grain's volume. In Table 2 the same is computed for different substrate and liquid pairs.

Table 1

Numerical values of S_b for $k = 2$ [13]

A [mm]	0.01	0.02	1		2	5		
θ	$\pi/40$	$\pi/40$	$\pi/12$	$\pi/11$	$\pi/11$	$\pi/40$	$\pi/12$	$\pi/11$
S_b [mm^2]	$2.5 \cdot 10^{-4}$	$1.06 \cdot 10^{-3}$	2.43	2.09	8.39	66.18	60.73	52.48

Table 2

Capillary - bridge - cohesive energy E for ellipsoidal grains; $\alpha = 10^{-3}$ m, $k = 2$, $V_K = 10^{-2}$ of the grain's volume

substrate-liquid	γ [10^{-2} J/m^2]	θ [°]	$E = \gamma S_b$ [J]
quartz-water	7.28	0	$13.68 \cdot 10^{-8}$
zinc-mercury	48.0	10	$90 \cdot 10^{-8}$
polymethacrylate methyl-water	7.28	56	$13.90 \cdot 10^{-8}$

In conclusion we computed the cohesive energy between ellipsoidal grains due to the small amount of liquid. We approximate free - liquid surface of the bridge by an toroid for (a) grain arrangement, see Figure 1. For (b) and (c) arrangements the toroidal approximations is not valid and other approximation is needed.

References

[1] Jaeger H.M., Nagel S.R., Behringer R.P., Granular solids, liquids and gases, Rev. Mod. Phys. 1996, 68, 4, 1259-1273.

[2] Albert R., Albert I., Hornbaker D., Schiffer P., Barabasi A.L., Maximum angle of stability in wet and dry spherical granular media, Phys. Rev. E 1997, 56, R6271-R6274.

[3] Barabasi A.L., Albert R., Schiffer P., The physics of sandcastles: maximum angle of stability in wet and dry granular media, Physica A 1999, 266, 366-371.

[4] Bocquet L., Restagno F., Charalix E., Where does a cohesive granular heap break? Eur. Phys. J.E. Soft Matter 2003, 14, 177-183.

[5] Bocquet L., Charlaix E., Restagno F., Physics of humid granular media, C.R. Physique 2002, 3, 207-215.

[6] Fourier Z., Geromichalos D., Herminghaus S., Kohonen M.M., Mugele F., Scheel M., Schulz M., Schulz B., Schier Ch., Seemann R., Skudelny A., Mechanical properties of wet granular materials, J. Phys.: Condens. Matter 2005, 17, 477, 5477-5502.

[7] Halsey T.C., Levine A.J., How sandcastles fall, Phys. Rev. Lett. 1998, 80, 3141-3144.

[8] Mason T.G., Levine A., Ertas D., Halsey T., The critical angle of wet sand piles, Phys. Rev. E 1999, 60, 5044-5047.

[9] Restagno F., Ursini C., Gayvallet H., Charlaix E., Aging in humid granular media, Phys. Rev. E 2002, 66, 021304-021311.

[10] Tegzes P., Albert R., Paskvan M., Barabasi A.L., Vicsek T., Schiffer P., Liquid-induced transitions in granular media, Phys. Rev. E 1999, 60, 5823-5826.

[11] Błaszczuk J., Domański Z., Liquid - induced glassy behaviour of dense non-spherical grain ensembles, Computer Methods in Mechanics Conference, Częstochowa, Poland 2005, 347-348.

[12] Han G., Rock stability under different fluid flow conditions, PhD thesis, Uniwersytet Waterloo, Ontario 2003.

[13] Błaszczuk J., Dynamics and statics of packing of granular materials, PhD thesis, Częstochowa 2005.