ON A LOWER SEMICONTINUITY OF A CERTAIN MULTIFUNCTION

Piotr Puchała

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. Let T be a metric space, X - n-dimensional Euclidean space and $P: T \to 2^X$ - continuous multifunction with compact convex values. We will show that multifunction $T \ni t \to extP(t) \in 2^X$ is lower semicontinuous.

1. Definitions

By T we will denote the metric space, by X - n-dimensional Euclidean space (although definitions and facts below can be stated in a more general setting).

1. We say that:

a) a set $A \subset X$ is convex, if whenever it contains two points, it also contains the line segment joining them; „algebraically speaking" A is convex, if $\lambda x + (1-\lambda) y \in A$ whenever $x, y \in A$ and $0 \le \lambda \le 1$;

b) a point $e \in A$ is an extreme point of A if and only if whenever $e = \lambda x + (1-\lambda) y,\ x, y \in A,\ 0 < \lambda < 1,$ then $x = y = e$ (by ext$A$ we will denote the set of extreme points of A);

c) the convex hull of $A \subset X$ (denoted by conv$A$) is the set of all convex combinations of points of A

$$cvA := \left\{ x : x = \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \ge 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

d) a point $e \in A$ is an exposed point of this set if there is an $f \in X^*$ such that $f(e) > f(x)$ whenever $e \ne x, x \in A$ (we then say that functional f exposes point e) (an exposed point $e \in A$ is always extreme (if not, there would exist $x, y \in A,\ \ x \ne e, y \ne e$ and number $\lambda \in (0,1)$ such that $e = \lambda x + (1-\lambda) y.$ On the other hand, by the linearity of the exposing functional f we would have $\lambda f(e) + (1-\lambda) = \lambda f(x) + (1-\lambda) f(y).$ But e is exposed, so $f(e) > f(x),$ $f(e) > f(y),$ a contradiction), but the converse need not be true even in $\Re^2$. For instance, let B denote the closed unit ball in the plane and $p \notin B$. Draw tangent lines from p to B, intersecting this at points x and y. Then the line segments px and py together with the larger arc xy of B form compact convex set C. Points x and y belong to $\operatorname{ext}C$, but they are not members of the set $\exp C$, the set of exposed points of C);

e) a slice of the set A determined by $f \in X^*$ and $\alpha > 0$ is the set

$$S(A, f, \alpha) := \{x \in A : f(x) > \sup\{f(x) : x \in A\} - \alpha\}$$

2. Denote by d metric generated by the norm in X. For the sets $A, B \subset X$ we define:
   a) $h^*(A,B) := \sup\{d(a,B) : a \in A\}$
   b) $h(A,B) := \max\{h^*(A,B), h^*(B,A)\}$
   The number $h(A,B)$ is called the Hausdorff distance between the sets A and B. The set of nonempty closed subsets of X with Hausdorff distance is a metric space.
3. The support function of a nonempty set $A \subset X$ is a function from X^* into $\Re \cup \{+\infty\}$ defined by $c(f, A) := \sup\{f(a) : a \in A\}$.

   A multifunction P is a mapping from the space T into nonempty subsets of a space X. Let $\varnothing \neq A \subset X$. We will use the following notation:

$$P^+(A) := \{x \in X : P(x) \subseteq A\}$$
$$P^-(A) := \{x \in X : P(x) \cap A \neq \varnothing\}$$

4. We say that multifunction $P : T \to 2^X - \{\varnothing\}$ is:
   a) lower semicontinuous, if the set $P^-(V)$ is open in T for every V open in X;
   b) upper semicontinuous, if the set $P^+(V)$ is open in T for every V open in X;
   c) continuous, if it is both lower- and upper semicontinuous.

2. Facts

In this section we state without proof more or less known facts which will be needed in further considerations.

1. (Krein-Milman theorem) A compact convex set $A \subset X$ is equal to the closed convex hull of its extreme points.
2. (Straszewicz theorem) A compact convex subset of $\Re^n$ is equal to the closed convex hull of its exposed points.
3. In a compact convex set $A \subset X$ the set $\exp A$ is dense in the set $\operatorname{ext} A$.
4. If A is as above and point $e \in A$ is exposed with exposing functional f, then from the fact that $f(x_n) \xrightarrow[n\to\infty]{} f(e)$ for arbitrary chosen sequence $(x_n) \subset A$ it

follows that $x_n \xrightarrow[n\to\infty]{} e$ in the topology of X (that is slices of A determined by f form a base of neighbourhoods of e in the relative topology of A).

5. If A, B are nonempty, closed and convex subsets of X, then

$$h(A,B) = \sup\{|c(f,A) - c(f,B)| : \|f\| \le 1\}$$

6. A multifunction $P: T \to 2^X - \{\varnothing\}$ is lower semicontinuous if and only if for every sequence $(t_n) \subset T$ and any point $x_0 \in P(t_0)$ there exists sequence $(x_n) \subset X$ convergent to x_0 and such that $x_n \in P(t_n)$.
7. A multifunction $P: T \to 2^X - \{\varnothing\}$ with compact values is upper semicontinuous if and only if for every $t \in T$, every sequence $(t_n) \subset T$ convergent to t, the sequence $(x_n) \subset X, x_n \in P(t_n)$ has a subsequence convergent to the limit belonging to $P(t)$.
8. A multifunction $P: T \to 2^X - \{\varnothing\}$ with compact values is continuous if and only if it is continuous in the Hausdorff metric.

3. Result

Let T be a metric space, $X - n$ dimensional Euclidean space, $P: T \to 2^X - \{\varnothing\}$ - continuous multifunction with compact convex values.
Then multifunction

$$t \to extP(t)$$

is lower semicontinuous.

Proof. Let (t_n) be a sequence in T, convergent to the point $(t_0) \in T$. We have to show, that for each such sequence and any $a_0 \in extP(t_0)$ there exists sequence $(a_n), a_n \in extP(t_n)$ such that $a_n \xrightarrow[n\to\infty]{} a_0$.

Let e_0 be any exposed point of $P(t_0)$. Then there exists functional $f_0 \in X^*$ with unit norm, exposing e_0. P is lower semicontinuous, so there exists sequence $(x_n) \subset X$, $x_n \xrightarrow[n\to\infty]{} e_0, x_n \in P(t_n)$.

Fix $\gamma > 0$ and define the set

$$R_\gamma(t_n) := \{x \in P(t_n) : f_0(x) > c(f_0, P(t_n)) - \gamma\}$$

Then there exists $n_0 \in N$ such that for every $n \ge n_0$ we have $R_\gamma(t_n) \cap extP(t_n) \ne \varnothing$. Suppose not. Then for each $n_0 \in N$ there exists $n \ge n_0$ for which

$R_\gamma(t_n) \cap extP(t_n) = \varnothing$, i.e. $extP(t_n) \subset X - R_\gamma(t_n)$. This yields existence of a subsequence $n_k \xrightarrow[k\to\infty]{} \infty$ with the following property: for each $e \in extP(t_{n_k})$ there holds an inequality

$$f_0(e) < c(f_0, P(t_{n_k})) - \gamma$$

By the Krein-Milman theorem $P(t_{n_k}) = clcvextP(t_{n_k})$, so for any $x \in P(t_{n_k})$ we have

$$f_0(x) \le c(f_0, P(t_{n_k})) - \gamma$$

In particular

$$f_0(x_{n_k}) \le c(f_0, P(t_{n_k})) - \gamma, \quad k = 1,2,....$$

Taking limits of the both sides we obtain

$$f_0(e_0) \le c(f_0, P(t_0)) - \gamma$$

a contradiction.

Now let $\gamma = \frac{1}{m}$, $m = 1,2,...$ and consider slices $R_{\frac{1}{m}}(\cdot)$. Then for each m there exists n_m such that for $n \ge n_m$ we have

$$R_{\frac{1}{m}}(t_n) \cap extP(t_n) \ne \varnothing$$

We can assume that $n_m \le n < n_{m+1}$. For such n choose

$$e_n \in R_{\frac{1}{m}}(t_n) \cap extP(t_n)$$

We have thus obtained a sequence (e_n) with the property

$$f_0(e_n) \ge c(f_0, P(t_n)) - \frac{1}{m}$$

for $n_m \le n < n_{m+1}$.

There exists a subsequence of the sequence (e_n) (denote it also by (e_n)), convergent to the point $\bar{e}_0 \in P(t_0)$. Continuity of P as well as compactness and convexity of its values then yield

$$\sup_{\|f\|\le 1}\left|c(f,P(t_n))-c(f,P(t_0))\right| = h(P(t_n),\ P(t_0)) \xrightarrow[n\to\infty]{} 0$$

Thus $c(f,P(t_n)) \xrightarrow[n\to\infty]{} c(f,P(t_0))$. But for $n_m \le n < n_{m+1}$ we have $f_0(e_n) \ge c(f_0,P(t_n)) - \frac{1}{m}$; we also have $f_0(e_n) \xrightarrow[n\to\infty]{} f_0(\bar{e}_0)$, so $f_0(\bar{e}_0) \ge c(f_0,P(t_0))$ and finally $\bar{e}_0 = e_0$.

Now by the Facts 2 and 3 we have that the set $P(t_0)$ is equal to the closed convex hull of its exposed points and that those points form a dense subset in $extP(t_0)$.

Now let a_0 be any point of $extP(t_0)$. Choose and fix $n_1 \in N$ and $e_0^{n_1} \in \exp P(t_0)$. There exists sequence (b_n^1), $b_n^1 \in extP(t_0)$, convergent to $e_0^{n_1}$. Then there exists $n_2 > n_1$ such that for $n \ge n_2$ we have $\left\|e_0^{n_1} - b_n^1\right\| < \frac{1}{n_1}$. Now take $e_0^{n_2} \in \exp P(t_0)$ with $\left\|e_0^{n_2} - a_0\right\| < \frac{1}{n_2}$ and sequence (b_n^2), $b_n^2 \in extP(t_n)$, convergent to $e_0^{n_2}$. Then there exists $n_3 > n_2$ such that for $n \ge n_3$ we have $\left\|e_0^{n_2} - b_n^2\right\| < \frac{1}{n_2}$. Continuing this way we obtain sequences (b_n^i), $b_n^i \in extP(t_n)$. Putting $a_n := b_n^i$ for $n_i \le n < n_{i+1}$ we finish constructing of a desire sequence (a_n), $a_n \in extP(t_n)$, $a_n \xrightarrow[n\to\infty]{} a_0$. This proves that multifunction $extP(t_0)$ is lower semicontinuous.

Remark

A.A. Tolstonogov and A.I. Figonienko proved the same result in a more general setting- with *X* being a Banach space. However they used quite different, topological methods. The proof presented here seems to be appealing to the intuition by its geometric character. Moreover, it seems possible to use this method in proving the result for a Banach space and even in proving a generalization of this result, namely, with multifunction *P* having only closed convex values.

Acknowledgments

The author would like to thank prof. A. Fryszkowski for numerous conversations.

References

[1] Bourgin R.D., Geometric aspects of convex sets with Radon-Nikodym property, Springer Lecture Notes in Mathematics 993, 1983.

[2] Repovs D., Semenov P.V., Continuous selections of multivalued mappings, Kluwer 1998.

[3] Tolstonogov A.A., Figonienko A.I., On functional-differential inclusions with non-convex right--hand side in a Banach space, Dokl. AN. SSSR 254 (1980), 45-49 (in Russian).

[4] Webster R., Convexity, Oxford University Press, 1994.