BOUNDARY ELEMENT METHOD IN THE INVERSE PROBLEMS OF STEADY HEAT TRANSFER

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Abstract. The methods of inverse problems solution appearing in the domain of steady heat transfer are discussed. In particular, the inverse boundary problems (the identification of the boundary values on the part of the surface limiting the system analyzed) and the inverse parametric problems (reconstruction of the thermophysical parameters of the material) are considered. Such problems have been solved using (on the stage of numerical computations) the boundary element method. The different methods of inverse problems solution have been applied. So, the direct method, the least squares method in the basic version, the same method supplemented by the regularization terms, the method of the energy minimization and also the algorithm basing on the sensitivity coefficients have been taken into account. The computations can been realized for different numbers and positions of the control points, the possible disturbances of 'measured' temperatures have been also taken into account. The remarks concerning the exactness and effectiveness of successive methods of the inverse problem solution have been formulated. The theoretical considerations are supplemented by the examples of computations verifying the correctness of the algorithms proposed.

1. Inverse boundary problems

The inverse boundary problems concern the identification of the boundary condition on the part Γ_1 of the surface Γ limiting the system analyzed [1-7]. The unknown quantity on Γ_1, this means the temperature (Dirichlet condition), heat flux (Neumann condition) - Figure 1, or heat transfer coefficient in the Robin condition can be determined under the assumption that the additional information concerning the values of temperature at the set of internal points from the domain considered is also given.

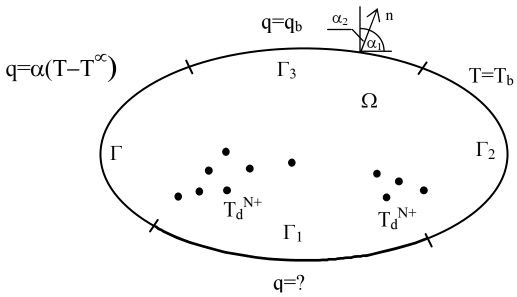

Fig. 1. Identification of boundary heat

As an example, let us consider the following 2D inverse boundary problem

$$\begin{cases} x \in \Omega : & \nabla^2 T(x) = 0 \\ x \in \Gamma_1 : & q(x) = -\lambda\, \partial T(x)/\partial n = ? \\ x \in \Gamma_2 : & T(x) = T_b \\ x \in \Gamma_3 : & q(x) = q_b \\ x \in \Gamma_4 : & q(x) = \alpha\,[T(x) - T^{\infty}] \\ \xi^{\mathrm{i}} \in \Omega : & T_d^i - \text{known}, \quad i = N+1, N+2, ..., N+M \end{cases} \tag{1}$$

where T_b is the given boundary temperature, q_b is the known boundary heat flux, $\partial T/\partial n$ denotes the normal derivative at the boundary point x, α is the heat transfer coefficient, T^{∞} is the ambient temperature. The aim of investigations is to determine the boundary heat flux on Γ_1.

In order to solve the problem considered the least squares criterion, as a rule, is applied, e.g. [1, 5-7]

$$S = \sum_{i=N+1}^{N+M} \left(\mathrm{T}^{\mathrm{i}} - T_d^i\right)^2 \tag{2}$$

where T^i is the calculated value of temperature at the internal point ξ^{i}, T_d^i is the know (e.g. resulting from measurements) temperature at the same internal point. The basic sum of squares can be supplemented by the regularization term [5, 6]

$$S = \sum_{i=N+1}^{N+M} \left(T^i - T_d^i\right)^2 + \gamma \sum_{k=1}^{N_1} q_k^2 \tag{3}$$

where γ is the regularization parameter, q_k is the unknown heat flux at the boundary point $x^k \in \Gamma_1$, N_1 is the number of points on boundary Γ_1. The solution of inverse problem consists in the searching of functional (2) or (3) minimum.
If the energy minimization method is used, then the minimum of functional [4]

$$S = -\frac{1}{2\lambda} \int_{\Gamma} T\, q \,\mathrm{d}\Gamma \tag{4}$$

with the following restrictions

$$\left| T^i - T_d^i \right| < \varepsilon, \qquad i = N+1, N+2, \dots, N+M \tag{5}$$

should be determined. In equation (5) ε is a certain small number.

2. Application of the BEM in the steady heat transfer problems

The boundary integral equation for the Laplace problem is of the form [5, 6, 9, 10]

$$B(\xi)\,T(\xi)+\int_\Gamma T^*(\xi,x)\,q(x)\,\mathrm{d}\Gamma=\int_\Gamma q^*(\xi,\ x)\,T(x)\,\mathrm{d}\Gamma \tag{6}$$

where $\xi \in \Gamma$ is the observation point, $B(\xi)\in(0,\ 1)$, $T^*(\xi,\ x)$ is the fundamental solution [5, 6, 9, 10] and $q^*(\xi,x)=-\lambda\,\partial T^*(\xi,x)/\partial n$.

In numerical realization, the boundary Γ is divided into N constant boundary elements Γ_j. Additionally, we assume that N_1 nodes belong to the boundary Γ_1, the nodes $N_1+1,\dots,\ N_2$ belong to Γ_2, the nodes $N_2+1,\dots,\ N_3$ belong to Γ_3, while nodes $N_3+1,\dots,\ N$ - to the boundary Γ_4. The integrals in equation (6) are substituted by sum of integrals and then for constant boundary elements one obtains ($i=1,\dots,\ N$)

$$\xi^i\in\Gamma:\quad \sum_{j=1}^{N} G_{ij}q_j=\sum_{j=1}^{N} H_{ij}T_j \tag{7}$$

where

$$G_{ij}=\int_{\Gamma_j} T^*(\xi^i,x)\,\mathrm{d}\Gamma_\mathrm{j} \tag{8}$$

and

$$H_{ij}=\begin{cases}\int_{\Gamma_j} \mathrm{q}^*(\xi^i,x)\,\mathrm{d}\Gamma_j, & i\neq j\\ -0.5, & i=j\end{cases} \tag{9}$$

while $T_j=T(x^j)$, $q_j=q(x^j)$. The temperatures at internal nodes ($i=N+1,\dots,\ N+M$) are calculated using the formula

$$\xi^i\in\Omega:\quad T^i=T(\xi^i)=\sum_{j=1}^{N} H^w_{ij}\,T_j-\sum_{j=1}^{N} G^w_{ij}\,q_j \tag{10}$$

where

$$\xi^i\in\Omega:\quad G^w_{ij}=\int_{\Gamma_j} T^*(\xi^i,x)\,\mathrm{d}\Gamma_j \tag{11}$$

and

$$\xi^i\in\Omega:\quad H^w_{ij}=\int_{\Gamma_j} q^*(\xi^i,x)\,\mathrm{d}\Gamma_j \tag{12}$$

3. Algorithms of inverse boundary problems solution

Taking into account the boundary conditions (1), the system of equations (7) can be written as follows

$$
\begin{gathered}
-\sum_{j=1}^{N_1} H_{ij} T_j + \sum_{j=N_1+1}^{N_2} G_{ij} q_j - \sum_{j=N_2+1}^{N_3} H_{ij} T_j + \sum_{j=N_3+1}^{N} (\alpha G_{ij} - H_{ij}) T_j = \\
= -\sum_{j=1}^{N_1} G_{ij} q_j + \sum_{j=N_1+1}^{N_2} H_{ij} T_b - \sum_{j=N_2+1}^{N_3} G_{ij} q_b + \sum_{j=N_3+1}^{N} \alpha G_{ij} T^{\infty}
\end{gathered}
\tag{13}
$$

or in the matrix form

$$
\mathbf{B_1}\, \mathbf{Y} = \mathbf{B_2}\, \mathbf{P} \tag{14}
$$

where

$$
\mathbf{Y} = [T_1 \dots T_{N_1} \;\; q_{N_1+1} \dots q_{N_2} \;\; T_{N_2+1} \dots T_{N_3} \;\; T_{N_3+1} \dots T_N]^{\mathrm{T}} \tag{15}
$$

and

$$
\mathbf{P} = [q_1 \dots q_{N_1} \;\; T_b \dots T_b \;\; q_b \dots q_b \;\; T^{\infty} \dots T^{\infty}]^{\mathrm{T}} \tag{16}
$$

The form of matrixes $\mathbf{B}_1$, $\mathbf{B}_2$ is presented in [10]. It should be pointed out that the vector $\mathbf{P}$ contains the unknown boundary heat fluxes $q_1, q_2, \dots, q_{N1}$.
From the system (14) results that

$$
\mathbf{Y} = \mathbf{B_1^{-1}}\, \mathbf{B}_2\, \mathbf{P} = \mathbf{U}\, \mathbf{P} \tag{17}
$$

Using the formulas (15), (17) one obtains

$$
T_j = \sum_{k=1}^{N_1} U_{jk} q_k + \sum_{k=N_1+1}^{N} U_{jk} P_k, \quad j = 1, 2, \dots, N_1 \tag{18}
$$

and

$$
q_j = \sum_{k=1}^{N_1} U_{jk} q_k + \sum_{k=N_1+1}^{N} U_{jk} P_k, \quad j = N_1 + 1, N_1 + 2, \dots, N_2 \tag{19}
$$

while

$$
q_j = \alpha\,(T_j - T^{\infty}) = \alpha\,(T_j - P_j), \quad j = N_3 + 1, \dots, N \tag{20}
$$

The dependencies (18), (19), (20) are introduced into equations (10) and then

$$T^i = \sum_{j=1}^{N_1} W_{ij} q_j + Z_i \tag{21}$$

where

$$W_{ij} = -G_{ij}^w + D_{ij}^w \tag{22}$$

and

$$D_{ij}^w = \sum_{k=1}^{N_1} H_{ik}^w U_{kj} - \sum_{k=N_1+1}^{N_2} G_{ik}^w U_{kj} + \sum_{k=N_2+1}^{N_3} H_{ik}^w U_{kj} + \sum_{k=N_3+1}^{N} (H_{ik}^w - \alpha G_{ik}^w) U_{kj} \tag{23}$$

while

$$Z_i = \sum_{j=N_1+1}^{N_2} H_{ij}^w P_j - \sum_{j=N_2+1}^{N_3} G_{ij}^w P_j + \sum_{j=N_3+1}^{N} \alpha G_{ij}^w P_j + \sum_{j=N_1+1}^{N} D_{ij}^w P_j \tag{24}$$

If the *direct method* of inverse problem solution is applied, then the number of internal points ξ^i, in which the temperature $T_d^i = T_d(\xi^i)$ is known must be equal to the number of boundary nodes in which the heat fluxes are unknown, this means $M = N_1$. Using the formula (21) one obtains the following system of equations

$$\sum_{j=1}^{N_1} \mathrm{W}_{\mathrm{ij}} q_j + Z_i = T_d^i, \quad \mathrm{i} = 1, 2, ..., N_1 \tag{25}$$

or in the matrix form

$$\mathbf{W}\,\mathbf{q} = \mathbf{T}_{\mathrm{d}} - \mathbf{Z} \tag{26}$$

This system allows to determine the values of boundary heat fluxes q_j, $j = 1,..., N_1$.

In the case of *least squares method* application, the formula (21) is introduced into (2) (or into (3)) and next using the necessary condition of minimum of several variables function one has

$$\sum_{i=1}^{M} \sum_{j=1}^{N_1} W_{il} W_{ij} q_j = \sum_{i=1}^{M} (T_d^i - Z_i) W_{il}, \quad l = 1, 2, ..., N_1 \tag{27}$$

or in the matrix form

$$\mathbf{W}^{\mathrm{T}} \mathbf{W}\,\mathbf{q} = \mathbf{W}^{\mathrm{T}} (\mathbf{T}_d - \mathbf{Z}) \tag{28}$$

In the case of *energy minimization method* the minimum of functional (4) (after the discretization of the boundary Γ), corresponds to the minimum of following function [10]

$$S = -\sum_{j=1}^{N} T_j q_j \tag{29}$$

Taking into account the given boundary conditions (2) one has

$$\begin{gathered} S = [q_1 \dots q_{N_1} \;\; T_b \dots T_b \;\; q_b \dots q_b \;\; \alpha(T_{N_3+1} - T^{\infty}) \dots \alpha(T_N - T^{\infty})]^{\mathrm{T}} \\ [T_1 \dots T_{N_1} \;\; q_{N_1+1} \dots q_{N_2} \;\; T_{N_2+1} \dots T_{N_3} T_{N_3+1} \dots T_N] \end{gathered} \tag{30}$$

or using the equation (17)

$$S = -\mathbf{C}^{\mathrm{T}} \; \mathbf{U} \, \mathbf{P} \tag{31}$$

where (c.f. equation (18))

$$\begin{aligned} \mathbf{C}^{\mathrm{T}} = \Bigg[& q_1 \dots q_{N_1} \;\; T_b \dots T_b \;\; q_b \dots q_b \\ & \alpha\left(\sum_{k=1}^{N_1} U_{N_3+1,k} \, q_k + \sum_{k=N_1+1}^{N} U_{N_3+1,k} \, P_k - T^{\infty} \right) \dots \\ & \alpha\left(\sum_{k=1}^{N_1} U_{Nk} \, q_k + \sum_{k=N_1+1}^{N} U_{Nk} \, P_k - T^{\infty} \right) \Bigg] \end{aligned} \tag{32}$$

So, the energy minimization method leads to the solution of problem

$$\begin{cases} \min S(q_1, q_2, \dots, q_{N_1}) = \min\left(-\mathbf{C}^{\mathrm{T}} \cdot \mathbf{U} \cdot \mathbf{P}\right) \\ \left| \sum_{j=1}^{N_1} W_{ij} \, q_j - Z_i \right| \le \varepsilon, \quad i = 1, 2, \dots, M \end{cases} \tag{33}$$

where W_{ij}, Z_i are described by formulas (22) and (24).

The algorithm of unknown boundary heat flux identification constructed on the basis of the *least squares criterion* (3) in which the *sensitivity coefficients* are introduced is the following [5]. At first, we solve the basic boundary problem for the arbitrary assumed values of local heat fluxes along the boundary Γ_1, for instance $q_k = 0$ for $k = 1, 2, \dots, N_1$. The solution obtained we denote by T^*, q^* (temperatures and heat fluxes). The function T is expanded into Taylor's series in the vicinity of point T^{*i} taking into account the first and second components, this means

$$T^i = T^{*i} + \sum_{k=1}^{N_1} R_k^i (q_k - q_k^*) \tag{34}$$

where

$$R_k^i = \left(\frac{\partial T}{\partial q_k} \right)_{x=x^i} \tag{35}$$

are the sensitivity coefficients [5, 10].

In order to determine the sensitivity coefficients the governing equations (1) should be differentiated with respect to q_k, $k = 1, 2,..., N_1$, namely

$$\begin{cases} x \in \Omega & : \ \nabla^2 R_k(x) = 0 \\ x \in \Gamma_1 & : \ V_k(x) = \begin{cases} 1 \ , \ x = x^k \\ 0 \ , \ x \neq x^k \end{cases} \\ x \in \Gamma_2 & : \ R_k(x) = 0 \\ x \in \Gamma_3 & : \ V_k(x) = 0 \\ x \in \Gamma_4 & : \ V_k(x) = \alpha R_k(x) \end{cases} \tag{36}$$

where $V_k(x) = -\lambda\, \partial Z_k(x)/\partial n$. One can notice that the problems described by (36) are correctly posed and should be treated as a direct one. So, we can use the same algorithm as in chapter 2 and in this way to find the set of sensitivity coefficients at internal points ξ^i for which the temperatures are known (measured) - see Figure 1. These coefficients are collected in the matrix **R**

$$\mathbf{R} = \begin{bmatrix} R_1^{N+1} & R_2^{N+1} & \dots & R_k^{N+1} & \dots & R_{N_1}^{N+1} \\ R_2^{N+2} & R_2^{N+2} & \dots & R_k^{N+2} & \dots & R_{N_1}^{N+2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ R_1^{N+M} & R_2^{N+M} & \dots & R_k^{N+M} & \dots & R_{N_1}^{N+M} \end{bmatrix} \tag{37}$$

We put (34) into (3). Next we differentiate the criterion (3) with respect to the unknown heat fluxes q_l, $l = 1, 2,..., N_1$ and using the necessary condition of minimum we obtain

$$\sum_{i=N+1}^{N+M} \sum_{k=1}^{N_1} R_k^i R_l^i q_k + \gamma\, q_l = \sum_{i=N+1}^{N+M} \left[R_l^i (T_d^i - T^{*i}) + \sum_{k=1}^{N_1} R_k^i R_l^i q_k^* \right] \tag{38}$$

The system of equations (36) can be written in the matrix form, namely

$$(\mathbf{R}^{\mathrm{T}}\mathbf{R} + \gamma \mathbf{I})\, \mathbf{q} = \mathbf{R}^{\mathrm{T}}(\mathbf{T_d} - \mathbf{T}^*) + \mathbf{R}^{\mathrm{T}}\mathbf{R}\, \mathbf{q}^* \tag{39}$$

where **I** is the identity matrix.

4. Inverse parametric problems

Let us consider the problem of thermal conductivity identification. This parameter can be treated as a constant value but, as a rule, it is the temperature dependent function. The thermal conductivity is determined on the basis of physical experiments. From the mathematical point of view the identification of this parameter on the basis of the knowledge of temperature field in the domain considered belongs to the group of the parametric inverse problems [2, 5].

As an example, the steady temperature field in domain Ω is analyzed

$$x \in \Omega: \quad \nabla [\lambda(T) \nabla T(x)] = 0 \tag{40}$$

where λ is the temperature dependent thermal conductivity

$$\lambda(T) = a\,T^2 + b\,T + c \tag{41}$$

while the coefficients a, b, c are unknown. On the boundary Γ the Dirichlet condition in the form

$$x \in \Gamma: \quad T(x) = T_b \tag{42}$$

is accepted. Additionally, it is assumed, that the value of thermal conductivity for temperature T_d is known, namely $\lambda_d = \lambda(T_d)$ and also two temperatures at internal points, this means $T_d^1 = T(x^1)$ and $T_d^2 = T(x^2)$ are given. The aim of investigations is to determine the values of a, b, c in the equation (41). The algorithm of the problem considered solution basing on the BEM is presented in [10].

5. Examples of computations

The problem of identification of heat flux between casting and continuous casting mould (CCM) will be presented. We consider the symmetrical fragment of CCM shown in Figure 2. The thickness of CCM equals 0.05 m, diameter of cooling pipe equals 0.02 m. The thermal conductivity: $\lambda = 330$ W/mK. In Figure 2 the boundary conditions and also the temperatures at the internal nodes of CCM are shown. These temperatures correspond approximately to the temperatures obtained from the direct problem solution under the assumption that the heat flux between casting and CMM equals $-3 \cdot 10^5$ W/m^2. In Figure 3 the discretization of the boundary is presented.

In order to identify the boundary heat flux the least squares criterion in which the sensitivity coefficients are introduced has been applied (c.f. equation (37)). So, five additional problems connected with the sensitivity analysis of temperature field with respect to q_1, q_2, q_3, q_4, q_5 have been solved. In Figures 4 and 5 the distributions of sensitivity functions R_1 and R_5 are shown.

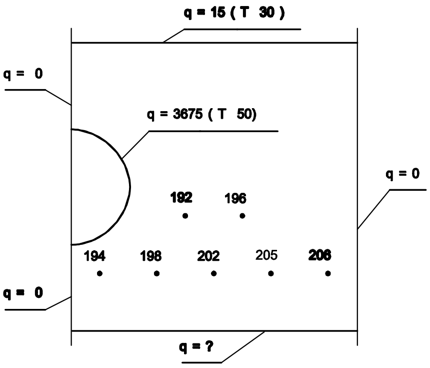

Fig. 2. Domain considered

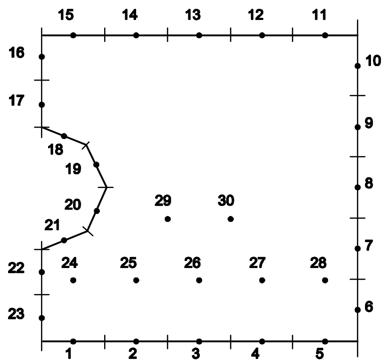

Fig. 3. Discretization

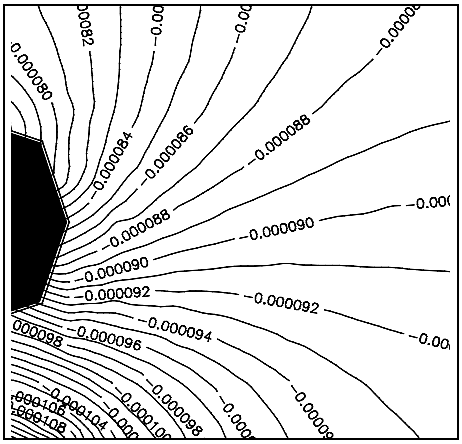

Fig. 4. Distribution of function R_1

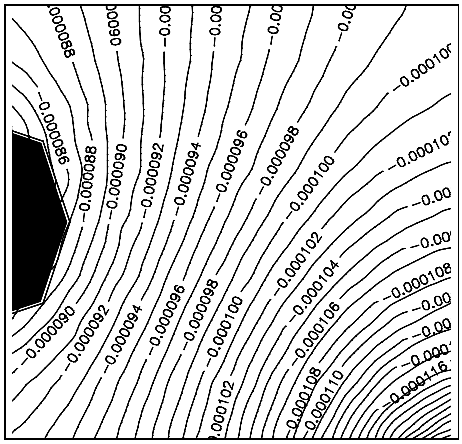

Fig. 5. Distribution of function R_2

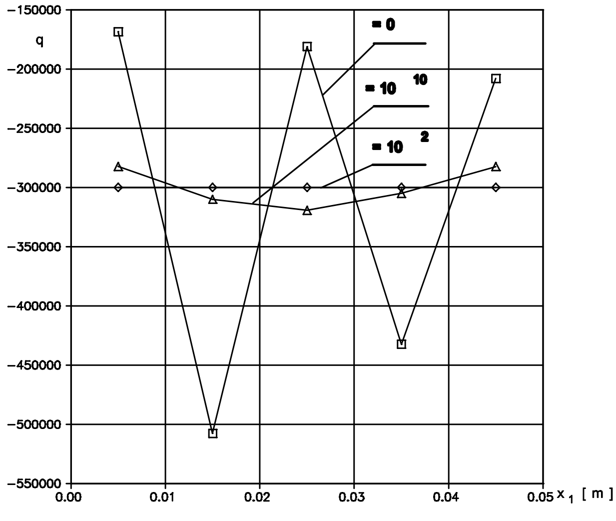

Fig. 6. Identified values of boundary heat flux for different values of parameter γ

The best solution of the inverse problem has been obtained for regularization parameter $\gamma = 10^{-2}$ (Fig. 6) and then we find

$$q_1 = q_2 = q_3 = q_4 = q_5 = -299999.995 \tag{43}$$

Summing up, the least squares criterion with the sensitivity coefficients and regularization parameter leads to the exact and efficient algorithm of the boundary heat flux identification.

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