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ASYMPTOTIC ANALYSIS OF FLEXURAL, TORSIONAL, AND STRETCHING MODES IN A MULTI-ROD STRUCTURE

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Abstract. We investigate the asymptotic behavior of an eigenvalue problem arising in a multi-rod structure with clamped ends as the rod thickness tends to zero. For low-frequency modes, we prove convergence of the eigenvalues and eigenfunctions toward those of the classical flexural limit model. For higher-frequency modes, we introduce a spectral analysis framework and show that the associated eigenfunctions admit limit descriptions in terms of torsional and stretching vibrations, depending on the considered sequences.

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1. Introduction

The asymptotic analysis of eigenvalue problems in thin elastic structures is a fundamental topic in mathematical physics and engineering. Understanding the limiting behavior of eigenmodes as the thickness of these structures approaches zero is essential for rigorously justifying reduced-dimensional models and accurately capturing key mechanical effects. The analysis of eigenvalue problems in plates and rods has a rich and well-established history, with early foundational work by Ciarlet & Kesavan [1] establishing rigorous approximations of three-dimensional elasticity problems in plate theory.

A significant amount of research has been devoted to the low-frequency modes of thin elastic structures. In particular, Le Dret [2] analyzed the eigenvalue problem for folded plates, while Kerdid [3] extended the study to the eigenvalue problem for multi-rod structures.

We also highlight several relevant studies on the asymptotic analysis of eigenvalue problems in various thin linear elastic structures. Notably, Jimbo & Rodríguez Mulet [4] and Jimbo et al. [5] investigated thin elastic rods with non-uniform cross-sections,

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Bunoiu et al. [6] analyzed the junctions of thin rods and a plate, Tambavca [7] focused on curved rods, and Gaudiello et al. [8, 9] studied eigenmodes in thin T-like shaped structures and in a multi-structure composed of two joined perpendicular thin films. Furthermore, Nazarov et al. [10] developed a one-dimensional asymptotic model for an L-shaped junction of two thin elastic beams, while Leugering et al. [11] described the asymptotic behavior of planar elastic beams connected by flexible joints. All these works aim to rigorously justify one-dimensional or two-dimensional approximations of higher-dimensional problems.

While low-frequency eigenmodes are well understood and often described by classical flexural displacement models, the characterization of high-frequency modes remains a more intricate problem and requires refined asymptotic techniques. In this work, we analyze the eigenmodes of a three-dimensional eigenvalue problem in linear elasticity for a multi-structure consisting of two thin rods meeting at a right angle. The case of low-frequency modes where the multi-structure is assumed to be clamped at one of its ends only was studied in [3]. Following the same arguments, we establish the convergence of low-frequency modes, demonstrating that the eigenvalues and eigenfunctions converge to those of a well-posed, one-dimensional limiting problem governed by coupled fourth-order differential equations with appropriate junction conditions. This confirms that low-frequency modes primarily correspond to flexural displacements in each rod. The limiting junction relations express the fact that the multi-structure remains in one piece and that the rod axes, after deformation, remain at a right angle at the junction. They also show that each rod undergoes torsions that follow the deformations of the end of the other rod at the junction.

The main challenge, however, lies in the characterization of high-frequency modes. Building on previous works in asymptotic analysis for rods [12,13] and plates [14,15] we introduce a spectral analysis technique that allows us to characterize the high-frequency eigenvalues and eigenfunctions, as the thickness of the rods vanishes, by associating a family of indices that depend on the thickness parameter. These modes ultimately converge to the torsional and stretching vibrations of the rods. The associated limiting displacements $(\bar{u}^{1,m}, \bar{u}^{2,m})$ are of the form:

$$\bar{u}^{1,m}(x) = (\zeta^{1,m}(x_1), (x_3 - 1/2)\theta^{1,m}(x_1), -(x_2 - 1/2)\theta^{1,m}(x_1)),$$

$$\bar{u}^{2,m}(x) = ((x_3 - 1/2)\theta^{2,m}(x_2), \zeta^{2,m}(x_2), -(x_1 - 1/2)\theta^{2,m}(x_2)),$$

with
$$\zeta^{1,m}, \zeta^{2,m}, \theta^{1,m}, \theta^{2,m} \in H^1([0,1]). \quad (1)$$

The pair $(\zeta^{1,m}, \zeta^{2,m})$ corresponds to the stretching displacements of the rods while $(\theta^{1,m}, \theta^{2,m})$ corresponds to the torsional ones. They verify, respectively, the system of classical equations of stretching vibrations and the system of classical equations of torsional vibrations.

The structure of this paper is as follows: In Section 2, we present the three--dimensional eigenvalue problem for the multi-rod structure. In Section 3, we establish the convergence of low-frequency modes and derive the corresponding one-dimensional limiting problem. In Section 4, we introduce the asymptotic framework necessary for analyzing high-frequency modes, and we present the main results concerning their convergence, linking them to torsional and stretching vibrations in the limiting model. Finally, Section 5 discusses the implications of our findings and outlines possible extensions to more complex multi-rod configurations.

2. The three-dimensional problem

Consider a multi-structure Ω^ε (Fig. 1) consisting of two thin rods of thickness $\varepsilon > 0$ meeting at a right angle, defined as

$$\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon$$

with

$$\Omega_1^\varepsilon = \left\{x \in \mathbb{R}^3 \mid 0 < x_1 < 1,\ 0 < x_2, x_3 < \varepsilon\right\},$$
$$\Omega_2^\varepsilon = \left\{x \in \mathbb{R}^3 \mid 0 < x_2 < 1,\ 0 < x_1, x_3 < \varepsilon\right\}.$$

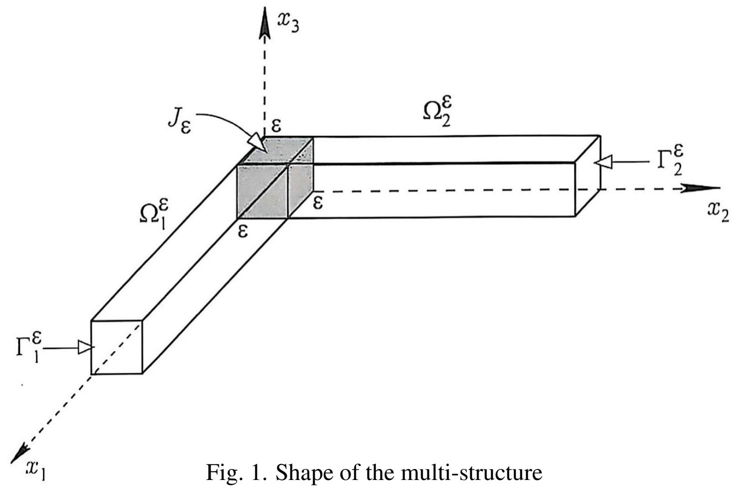

Fig. 1. Shape of the multi-structure

The junction region J^ε is given by

$$J^\varepsilon = \{x \in \Omega^\varepsilon\ ;\ 0 < x_1, x_2 < \varepsilon\}.$$

In this work, we consider the case where the multi-structure is clamped on both ends Γ_1^ε and Γ_2^ε defined as

$$\Gamma_1^\varepsilon = \partial\Omega_1^\varepsilon \cap \{x_1 = 1\}\ ,\ \Gamma_2^\varepsilon = \partial\Omega_2^\varepsilon \cap \{x_2 = 1\}.$$

The rod material is assumed to be homogeneous and isotropic, characterized by Lame's coefficients λ and μ.

In the sequel, we shall use the repeated index convention. Latin indices take their values in the set $\{1,2,3\}$, Greek indices with exponent 1 take their values in the set $\{2,3\}$, and Greek indices with exponent 2 take their values in the set $\{1,3\}$.

Understanding the free vibration modes of a structure is fundamental for understanding its dynamic behavior, particularly the asymptotic behavior as the thickness goes to zero. In this context, studying the associated eigenvalue problem provides a rigorous framework to analyse the limit vibration modes and identifying natural frequencies.

The eigenvalue problem for the multi-structure under consideration consists in finding pairs $(\eta^\varepsilon, u^\varepsilon)$ satisfying:

$$\begin{cases} -\partial_j \sigma_{ij}^\varepsilon & = \eta^\varepsilon u_i^\varepsilon & \text{in } \ \Omega^\varepsilon, \\ \sigma_{ij}^\varepsilon(u^\varepsilon) & = \lambda e_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(u^\varepsilon) & \text{in } \ \Omega^\varepsilon, \\ u^\varepsilon & = 0 & \text{on } \ \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon \\ \sigma^\varepsilon n^\varepsilon & = 0 & \text{on } \ \partial\Omega^\varepsilon \backslash \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon, \end{cases} \tag{2}$$

where $\sigma^\varepsilon(u^\varepsilon)$ is the stress tensor, n^ε is the outer unit normal vector to $\partial\Omega^\varepsilon$, and $e^\varepsilon(u^\varepsilon)$ is the linearized strain tensor corresponding to the displacement u^ε:

$$e_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_j^\varepsilon}{\partial x_i^\varepsilon} + \frac{\partial u_i^\varepsilon}{\partial x_j^\varepsilon}\right). \tag{3}$$

The variational formulation of problem (2) can be written:
Find $(\eta^\varepsilon, u^\varepsilon) \in \mathbb{R} \times V^\varepsilon$ satisfying

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon(u^\varepsilon) e_{ij}^\varepsilon(v^\varepsilon) dx^\varepsilon = \eta^\varepsilon \int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon dx^\varepsilon \ \ \forall v^\varepsilon \in V^\varepsilon, \tag{4}$$

where

$$V^\varepsilon = \left\{ v = (v_i) \in [H^1(\Omega^\varepsilon)]^3 \ v = 0 \ \text{ on } \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon \right\}. \tag{5}$$

Thanks to Korn inequality and the clamping conditions, problem (4) has a sequence of eigenvalues $(\eta_\ell^\varepsilon)_{\ell \geq 1}$ satisfying

$$0 < \eta_1^\varepsilon \leq \eta_2^\varepsilon \leq \eta_3^\varepsilon \leq \ldots \leq \eta_\ell^\varepsilon \leq \ldots \tag{6}$$

with

$$\lim_{\ell \longrightarrow \infty} \eta_\ell^\varepsilon = +\infty, \tag{7}$$

associated with a family of eigenfunctions $(u^{\varepsilon,\ell})_{\ell \geq 1}$, that is

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon(u^{\varepsilon,\ell}) e_{ij}^\varepsilon(v^\varepsilon) dx^\varepsilon = \eta_\ell^\varepsilon \int_{\Omega^\varepsilon} u_i^{\varepsilon,\ell} v_i^\varepsilon dx^\varepsilon \quad \forall v^\varepsilon \in V^\varepsilon. \tag{8}$$

These eigenfunctions can be orthonormalized as

$$\int_{\Omega^\varepsilon} u_i^{\varepsilon,m} u_i^{\varepsilon,n} dx^\varepsilon = \delta_{mn} \quad \forall m,n \geq 1, \tag{9}$$

and make a basis in both Hilbert spaces V^ε and $[L^2(\Omega^\varepsilon)]^3$.

The principal idea in studying the junction problems consists in scaling each part of the multi-structure independently of the other. Therefore, let us define

$$\Omega_1 = \Omega_1^1 \quad \Omega_2 = \Omega_2^1 \quad \Gamma_1 = \Gamma_1^1 \quad \Gamma_2 = \Gamma_2^1, \tag{10}$$

$$\omega_1 = \partial\Omega_1 \cap \{x_1 = 0\} \quad \omega_2 = \partial\Omega_2 \cap \{x_2 = 0\}, \tag{11}$$

and introduce the double change of scale

$$\begin{array}{rcl} \phi^\varepsilon \;:\; \Omega_1 \cup \Omega_2 & \to & \Omega^\varepsilon \\ x & \mapsto & \begin{cases} (x_1, \varepsilon x_2, \varepsilon x_3) & \text{if } x \in \Omega_1, \\ (\varepsilon x_1, x_2, \varepsilon x_3) & \text{if } x \in \Omega_2. \end{cases} \end{array}$$

The junction region is counted twice, and it is the image by ϕ^ε of the open sets

$$J_\varepsilon^1 = \{x \in \Omega_1 : x_1 < \varepsilon\} \;\text{ and }\; J_\varepsilon^2 = \{x \in \Omega_2 : x_2 < \varepsilon\}. \tag{12}$$

We define the space

$$V = H^1_{\Gamma_1}(\Omega_1, \mathbb{R}^3) \times H^1_{\Gamma_2}(\Omega_2, \mathbb{R}^3), \tag{13}$$

and we associate to this change of scale the operator:

$$\begin{array}{rcl} \Phi^\varepsilon \;:\; V^\varepsilon & \to & V \\ v^\varepsilon & \mapsto & ((v_1^\varepsilon, \varepsilon v_2^\varepsilon, \varepsilon v_3^\varepsilon) \circ \phi^\varepsilon, (\varepsilon v_1^\varepsilon, v_2^\varepsilon, \varepsilon v_3^\varepsilon) \circ \phi^\varepsilon). \end{array}$$

We denote

$$v(\varepsilon) = (v^1(\varepsilon), v^2(\varepsilon)) = \Phi^\varepsilon(v^\varepsilon). \tag{14}$$

The scaled displacements $(v^1(\varepsilon), v^2(\varepsilon))$ are then defined on two separate domains Ω_1 and Ω_2 and contain a double information on the junction region.
Let

$$V(\varepsilon) = \Phi^\varepsilon(V^\varepsilon). \tag{15}$$

It is easy to see that $V(\varepsilon)$ is the set of pairs $(v^1(\varepsilon), v^2(\varepsilon)) \in V$ satisfying the relations:

$$\begin{cases} \varepsilon v_1^1(\varepsilon)(\varepsilon x_1, x_2, x_3) = v_1^2(\varepsilon)(x_1, \varepsilon x_2, x_3), \\ v_2^1(\varepsilon)(\varepsilon x_1, x_2, x_3) = \varepsilon v_2^2(\varepsilon)(x_1, \varepsilon x_2, x_3), \\ v_3^1(\varepsilon)(\varepsilon x_1, x_2, x_3) = v_3^2(\varepsilon)(x_1, \varepsilon x_2, x_3). \end{cases} \tag{16}$$

These relations, known as the junction conditions, express the fact that in J_ε^1 and J_ε^2 the scaled eigen-displacements $(v^1(\varepsilon), v^2(\varepsilon))$ correspond to the same displacement of the multi-structure. They establish the necessary constraints that the solutions in Ω_1 and Ω_2 must satisfy to ensure a consistent displacement across the multi-structure.
Finally, we define the space of Bernoulli-Navier on Ω_1 as

$$V_{BN}(\Omega_1) = \left\{ v \in H^1(\Omega_1, \mathbb{R}^3),\; e_{\alpha^1 i}(v) = 0 \right\}. \tag{17}$$

Elements of this space are characterized by

$$\left\{\begin{array}{ll} v_1(x) & = \zeta_1(x_1) - (x_{\alpha^1} - 1/2)\zeta'_{\alpha^1}(x_1), \\ v_2(x) & = \zeta_2(x_1) + \theta(x_3 - 1/2), \\ v_3(x) & = \zeta_3(x_1) - \theta(x_2 - 1/2), \end{array}\right. \tag{18}$$

where $\zeta_{\alpha^1} \in H^2(0,1)$ and $\zeta_1 \in H^1(0,1)$, and $\theta \in \mathbb{R}$. We define $V_{BN}(\Omega_2)$ by the analogue formulas.

3. Convergence of the low frequency modes

The case of low-frequency modes in a multi-rod structure clamped at only one end was studied in [3]. Therefore, we will refer to this work whenever the proofs are identical.

We now introduce the following scaled bilinear form

$$\begin{array}{rcl} b^1_\varepsilon(u,v) & = & \varepsilon^{-4}[2\mu e_{\alpha^1\beta^1}(u)e_{\alpha^1\beta^1}(v) + \lambda e_{\alpha^1\alpha^1}(u)e_{\beta^1\beta^1}(v)] \\ & + & \varepsilon^{-2}[4\mu e_{\alpha^1 1}(u)e_{\alpha^1 1}(v) + \lambda(e_{\alpha^1\alpha^1}(u)e_{11}(v) + e_{11}(u)e_{\alpha^1\alpha^1}(v))] \\ & + & (\lambda + 2\mu)e_{11}(u)e_{11}(v), \end{array} \tag{19}$$

and a similar expression for $b^2_\varepsilon(u,v)$. Substituting (14) in (8) and (9), we obtain the following scaled variational formulation: Find $(\mu_\ell(\varepsilon), u^\ell(\varepsilon)) \in \mathbb{R} \times V(\varepsilon)$, such that for all $v(\varepsilon) \in V(\varepsilon)$

$$\begin{aligned} &\int_{\Omega_1} b^1_\varepsilon(u^{1,\ell}(\varepsilon), v^1(\varepsilon))\,dx + \int_{\Omega_2 \setminus J^2_\varepsilon} b^2_\varepsilon(u^{2,\ell}(\varepsilon), v^2(\varepsilon))\,dx \\ &\quad = \mu_\ell(\varepsilon)\int_{\Omega_1}\left[u^{1,\ell}_{\alpha^1}(\varepsilon)v^1_{\alpha^1}(\varepsilon) + \varepsilon^2 u^{1,\ell}_1(\varepsilon)v^1_1(\varepsilon)\right]dx \\ &\quad\quad + \mu_\ell(\varepsilon)\int_{\Omega_2 \setminus J^2_\varepsilon}\left[u^{2,\ell}_{\alpha^2}(\varepsilon)v^2_{\alpha^2}(\varepsilon) + \varepsilon^2 u^{2,\ell}_2(\varepsilon)v^2_2(\varepsilon)\right]dx. \end{aligned} \tag{20}$$

where

$$\mu_\ell(\varepsilon) = \varepsilon^{-2}\eta^\varepsilon_\ell, \tag{21}$$

and with the normalization condition

$$\begin{aligned} &\int_{\Omega_1}\left[u^{1,m}_{\alpha^1}(\varepsilon)u^{1,n}_{\alpha^1}(\varepsilon) + \varepsilon^2 u^{1,m}_1(\varepsilon)u^{1,n}_1(\varepsilon)\right]dx \\ &\quad + \int_{\Omega_2 \setminus J^2_\varepsilon}\left[u^{2,m}_{\alpha^2}(\varepsilon)u^{2,n}_{\alpha^2}(\varepsilon) + \varepsilon^2 u^{2,m}_2(\varepsilon)u^{2,n}_2(\varepsilon)\right]dx = \delta_{mn}. \end{aligned} \tag{22}$$

The scaled eigenvalues and eigenfunctions satisfy the following bounds:

Lemma 1 *For each $\ell \geq 1$, there exists constants δ_ℓ and $\bar{C}_\ell > 0$ independent of ε, such that*

$$\mu_\ell(\varepsilon) \leq \delta_\ell, \tag{23}$$

$$\| u^{\ell}(\varepsilon) \|_V \leq \bar{C}_{\ell}, \tag{24}$$

and we can extract, for each integer $\ell \geq 1$, a subsequence, still denoted ε, such that

$$\mu_{\ell}(\varepsilon) \to \mu_{\ell}(0) \qquad \text{in } \mathbb{R} \tag{25}$$

and
$$u^{\ell}(\varepsilon) \rightharpoonup u^{\ell}(0) \quad \text{weakly in } \mathrm{V}(\varepsilon) \tag{26}$$

where $u^{1,\ell}(0)$ and $u^{2,\ell}(0)$ are of Bernoulli-Navier type. □

PROOF The proof of (23) is exactly the same as in [3]. To establish (24), we choose $v(\varepsilon) = u^{\ell}(\varepsilon)$ in equation (20). Using the normalization condition (22), we deduce,

$$\begin{aligned} \|e_{\alpha^1\beta^1}(u^{1,\ell}(\varepsilon))\|_{L^2(\Omega_1)} &\leq C_{\ell}\varepsilon^2 \leq C_{\ell}, \\ \|e_{\alpha^1 1}(u^{1,\ell}(\varepsilon))\|_{L^2(\Omega_1)} &\leq C_{\ell}\varepsilon \leq C_{\ell}, \\ \|e_{11}(u^{1,\ell}(\varepsilon))\|_{L^2(\Omega_1)} &\leq C_{\ell}, \end{aligned} \tag{27}$$

and the similar bounds for $e_{ij}(u^{2,\ell^m_\varepsilon}(\varepsilon))$. Due to the Korn inequality and clamping conditions, we obtain (24) which yields (26). According to (26) and (27) we have,

$$e_{\alpha^1 i}(u^{1,\ell}(\varepsilon)) \rightharpoonup e_{\alpha^1 i}(u^{1,\ell}(0)) = 0.$$

Then $u^{1,\ell}(0) \in V_{BN}(\Omega_1)$, and the same for $u^{2,\ell}(0)$. ■

Since the limiting eigenfunctions are of Bernoulli-Navier type, there exists $\zeta^{1,\ell}_{\alpha^1}, \zeta^{2,\ell}_{\alpha^2} \in H^2([0,1])$, $\zeta^{1,\ell}_1, \zeta^{2,\ell}_2, \in H^1([0,1])$, *and* $\theta^{1,\ell}, \theta^{2,\ell} \in \mathbb{R}$ such that

$$\begin{cases} u^{1,\ell}_1(0)(x) &= \zeta^{1,\ell}_1(x_1) - (x_{\alpha^1} - 1/2)(\zeta^{1,\ell}_{\alpha^1})'(x_1), \\ u^{1,\ell}_2(0)(x) &= \zeta^{1,\ell}_2(x_1) + (x_3 - 1/2)\theta^{1,\ell}, \\ u^{1,\ell}_3(0)(x) &= \zeta^{1,\ell}_3(x_1) - (x_2 - 1/2)\theta^{1,\ell}, \end{cases} \tag{28}$$

and

$$\begin{cases} u^{2,\ell}_1(0)(x) &= \zeta^{2,\ell}_1(x_2) + (x_3 - 1/2)\theta^{2,\ell}, \\ u^{2,\ell}_2(0)(x) &= \zeta^{2,\ell}_2(x_2) - (x_{\alpha^2} - 1/2)(\zeta^{2,\ell}_{\alpha^2})'(x_2), \\ u^{2,\ell}_3(0)(x) &= \zeta^{2,\ell}_3(x_2) - (x_1 - 1/2)\theta^{2,\ell}. \end{cases} \tag{29}$$

Thanks to the clamping conditions, on $\Gamma_1 \cup \Gamma_2$ we have:

$$\zeta^{1,\ell}_i(1) = \zeta^{2,\ell}_i(1) = (\zeta^{1,\ell}_i)'(1) = (\zeta^{2,\ell}_i)'(1) = 0, \tag{30}$$

and

$$\theta^{1,\ell} = \theta^{2,\ell} = 0. \tag{31}$$

Lemma 2 *For each integer $\ell \geq 1$, the axial components of the limiting displacements vanish*

$$\zeta_1^{1,\ell} = \zeta_2^{2,\ell} = 0, \tag{32}$$

and the flexural components satisfy the following junction conditions:

$$\begin{cases} \zeta_2^{1,\ell}(0) &= \zeta_1^{2,\ell}(0) = 0, \\ \zeta_3^{1,\ell}(0) &= \zeta_3^{2,\ell}(0), \\ (\zeta_2^{1,\ell})'(0) &= -(\zeta_1^{2,\ell})'(0). \end{cases} \tag{33}$$

PROOF The proof of (32) is analogous to that in [3]. Relations (33) are obtained by passing to the limit in the junction conditions (16), adapting the techniques used in [16]. ■

Remark 1 The first relation in (33) highlights the rigidification effect induced by the junction on the two rods. The second relation describes the transmission of the vertical displacement components across the junction. Finally, the last relation ensures that, after deformation, the projections of the rod axes onto the plane remain perpendicular at the junction. □

Let us now characterize the limiting space of flexion displacements:

$$\begin{aligned} \mathcal{V}_F \;=\; & \{(\xi_{\alpha^1}^1, \xi_{\alpha^2}^2) \in H^2(0,1;\mathbb{R}^4) \,:\, \xi_{\alpha^1}^1(1) = (\xi_{\alpha^1}^1)'(1) = \xi_{\alpha^2}^2(1) = (\xi_{\alpha^2}^2)'(1) = 0, \\ & \xi_2^1(0) = \xi_1^2(0) = 0, \quad \xi_3^1(0) = \xi_3^2(0), \quad (\xi_2^1)'(0) = -(\xi_1^2)'(0)\}. \end{aligned} \tag{34}$$

Passing to the limit in the scaled variational formulation (20) and following the same steps as in [3] and [16], we can formulate the variational equation that governs the flexural displacements.

Theorem 1 *The flexural displacements $(\zeta_{\alpha^1}^{1,\ell}, \zeta_{\alpha^2}^{2,\ell})$ belong to $\mathcal{V}_F$ for each integer $\ell \geq 1$ and satisfy the following equation:* $\forall (\xi_{\alpha^1}^1, \xi_{\alpha^2}^2) \in \mathcal{V}_F$,

$$\begin{aligned} & E\int_0^1 I_{\alpha^1\beta^1}^1 (\zeta_{\alpha^1}^{1,\ell})''(x_1)(\xi_{\alpha^1}^1)''(x_1)dx_1 + E\int_0^1 I_{\alpha^2\beta^2}^2 (\zeta_{\alpha^2}^{2,\ell})''(x_2)(\xi_{\alpha^2}^2)''(x_2)dx_2 \\ & \quad + K^1(\zeta_3^{2,\ell})'(0)(\xi_3^2)'(0) + K^2(\zeta_3^{1,\ell})'(0)(\xi_3^1)'(0) \\ & \quad = \mu_\ell(0)\int_0^1 \zeta_{\alpha^1}^{1,\ell}(x_1)\xi_{\alpha^1}^1(x_1)dx_1 + \mu_\ell(0)\int_0^1 \zeta_{\alpha^2}^{2,\ell}(x_2)\xi_{\alpha^2}^2(x_2)dx_2. \end{aligned} \tag{35}$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{36}$$

is the Young's modulus, and

$$\begin{cases} I^1_{\alpha^1\beta^1} = \displaystyle\int_{\omega^1} \big(x_{\alpha^1} - \frac{1}{2}\big)\big(x_{\beta^1} - \frac{1}{2}\big) dx_2 dx_3, \\ I^2_{\alpha^2\beta^2} = \displaystyle\int_{\omega^2} \big(x_{\alpha^2} - \frac{1}{2}\big)\big(x_{\beta^2} - \frac{1}{2}\big) dx_1 dx_3, \end{cases} \tag{37}$$

$$\begin{cases} K^1 = 4\mu \displaystyle\int_{\omega^1} \|\nabla\chi^1\|^2 dx_2 dx_3, \\ K^2 = 4\mu \displaystyle\int_{\omega^2} \|\nabla\chi^2\|^2 dx_1 dx_3, \end{cases}$$

are the inertia moments and the torsional rigidity coefficients of each rod respectively with χ^1 and χ^2 the torsional functions defined on each section by

$$\begin{cases} \Delta\chi^1 = 1 & in \quad \omega^1, \\ \chi^1 = 0 & on \quad \partial\omega^1, \end{cases} \tag{38}$$

$$\begin{cases} \Delta\chi^2 = 1 & in \quad \omega^2, \\ \chi^2 = 0 & on \quad \partial\omega^2. \end{cases} \tag{39}$$

PROOF We follow the approach of [16], which involves constructing test functions that satisfy the three-dimensional junction relations (16), establishing convergence results, and then passing to the limit in the variational formulation (20) using these test functions.

More precisely, let ξ be an arbitrary element of $\mathscr{V}_F \cap \mathscr{C}^\infty(0,1;\mathbb{R}^4)$, and consider the development of its components to order 1 in the neighborhood of 0:

$$\xi^1_2(x_1) = x_1(\xi^1_2)'(0) + g^1_2(x_1), \qquad \xi^2_1(x_2) = x_2(\xi^2_1)'(0) + g^2_1(x_2),$$

$$\xi^1_3(x_1) = \xi^1_3(0) + x_1(\xi^1_3)'(0) + g^1_3(x_1), \qquad \xi^2_3(x_2) = \xi^2_3(0) + x_2(\xi^2_3)'(0) + g^2_3(x_2),$$

with the estimations

$$\sup\{|g^1_2(x_1)|, |g^1_3(x_1)|\} \leq Cx_1^2,$$
$$\sup\{|g^2_1(x_2)|, |g^2_3(x_2)|\} \leq Cx_2^2.$$

Let $v = (v_1, v_2)$ the displacement of Bernoulli-Navier be defined by

$$\begin{cases} v_1(x) = \Big(-\big(x_2 - \frac{1}{2}\big)(\xi^1_2)'(x_1) - \big(x_3 - \frac{1}{2}\big)(\xi^1_3)'(x_1),\ \xi^1_2(x_1),\ \xi^1_3(x_1) \Big), \\ v_2(x) = \Big(\xi^2_1(x_2),\ -\big(x_1 - \frac{1}{2}\big)(\xi^2_1)'(x_2) - \big(x_3 - \frac{1}{2}\big)(\xi^2_3)'(x_2),\ \xi^2_3(x_2) \Big). \end{cases} \tag{40}$$

We now define approximations of (v_1, v_2) that belong to $V(\varepsilon)$ as follows:

$$v_1^1(\varepsilon)(x) = \begin{cases} -(x_2 - \frac{1}{2})(\xi_2^1)'(0) - (x_3 - \frac{1}{2})(\xi_3^1)'(0), & 0 < x_1 < \varepsilon, \\ -(x_2 - \frac{1}{2})\big[(\xi_2^1)'(0) + (g_2^1)'(2(x_1 - \varepsilon))\big] & \\ \quad -(x_3 - \frac{1}{2})\big[(\xi_3^1)'(0) + (g_3^1)'(2(x_1 - \varepsilon))\big], & \varepsilon < x_1 < 2\varepsilon, \\ -(x_2 - \frac{1}{2})(\xi_2^1)'(x_1) - (x_3 - \frac{1}{2})(\xi_3^1)'(x_1), & 2\varepsilon < x_1 < 1. \end{cases}$$

$$v_2^2(\varepsilon)(x) = \begin{cases} -(x_1 - \frac{1}{2})(\xi_1^2)'(0) - (x_3 - \frac{1}{2})(\xi_3^2)'(0), & 0 < x_2 < \varepsilon, \\ -(x_1 - \frac{1}{2})\big[(\xi_1^2)'(0) + (g_1^2)'(2(x_2 - \varepsilon))\big] & \\ \quad -(x_3 - \frac{1}{2})\big[(\xi_3^2)'(0) + (g_3^2)'(2(x_2 - \varepsilon))\big], & \varepsilon < x_2 < 2\varepsilon, \\ -(x_1 - \frac{1}{2})(\xi_1^2)'(x_2) - (x_3 - \frac{1}{2})(\xi_3^2)'(x_2), & 2\varepsilon < x_2 < 1. \end{cases}$$

The tridimensional junction relations allow us to write: $\forall x_1, x_2 \in (0,1)$,

$$\begin{cases} v_2^1(\varepsilon)(\varepsilon x_1, x_2, x_3) = \varepsilon v_2^2(\varepsilon)(x_1, \varepsilon x_2, x_3) \\ \qquad\qquad = -\varepsilon(x_1 - \frac{1}{2})(\xi_1^2)'(0) - \varepsilon(x_3 - \frac{1}{2})(\xi_3^2)'(0), \\ v_1^2(\varepsilon)(x_1, \varepsilon x_2, x_3) = \varepsilon v_1^1(\varepsilon)(x_1, \varepsilon x_2, x_3) \\ \qquad\qquad = -\varepsilon(x_2 - \frac{1}{2})(\xi_2^1)'(0) - \varepsilon(x_3 - \frac{1}{2})(\xi_3^1)'(0). \end{cases}$$

So we can define $v_2^1(\varepsilon)$ and $v_1^2(\varepsilon)$ as follows:

$$v_2^1(\varepsilon)(x) = \begin{cases} -(x_1 - \frac{\varepsilon}{2})(\xi_1^1)'(0) - (x_3 - \frac{1}{2})(\xi_3^2)'(0), & 0 < x_1 < \varepsilon, \\ x_1(\xi_1^2)'(0) + (g_2^1)'(2(x_1 - \varepsilon)) & \\ \quad -\varepsilon\big[\frac{1}{2}(\xi_2^1)'(0) + (x_3 - \frac{1}{2})(\xi_3^2)'(0)\big], & \varepsilon < x_1 < 2\varepsilon, \\ \xi_2^1(x_1) - \frac{\varepsilon(1 - x_1)}{1 - 2\varepsilon}\big[\frac{1}{2}(\xi_2^1)'(0) + (x_3 - \frac{1}{2})(\xi_3^2)'(0)\big], & 2\varepsilon < x_1 < 1. \end{cases}$$

$$v_1^2(\varepsilon)(x) = \begin{cases} -(x_2 - \frac{\varepsilon}{2})(\xi_1^2)'(0) - (x_3 - \frac{1}{2})(\xi_3^1)'(0), & 0 < x_2 < \varepsilon, \\ x_2(\xi_2^1)'(0) + (g_1^2)'(2(x_2 - \varepsilon)) & \\ \quad -\varepsilon\big[\frac{1}{2}(\xi_1^2)'(0) + (x_3 - \frac{1}{2})(\xi_3^1)'(0)\big], & \varepsilon < x_2 < 2\varepsilon, \\ \xi_1^2(x_2) - \frac{\varepsilon(1 - x_2)}{1 - 2\varepsilon}\big[\frac{1}{2}(\xi_1^2)'(0) + (x_3 - \frac{1}{2})(\xi_3^1)'(0)\big], & 2\varepsilon < x_2 < 1. \end{cases}$$

For the third tridimensional junction relation to hold, we define the functions $v_3^1(\varepsilon)$ and $v_3^2(\varepsilon)$ as follows:

$$v_3^1(\varepsilon)(x) = \begin{cases} \xi_3^1(x_1) + \xi_3^2(\varepsilon x_2) - \xi_3^2(0), & 0 < x_1 < \varepsilon, \\ \xi_3^1(x_1) + \dfrac{1-x_1}{1-\varepsilon}\left[\xi_3^2(\varepsilon x_2) - \xi_3^2(0)\right], & \varepsilon < x_1 < 1, \end{cases}$$

$$v_3^2(\varepsilon)(x) = \begin{cases} \xi_3^2(x_1) + \xi_3^1(\varepsilon x_2) - \xi_3^1(0), & 0 < x_2 < \varepsilon, \\ \xi_3^2(x_1) + \dfrac{1-x_2}{1-\varepsilon}\left[\xi_3^1(\varepsilon x_2) - \xi_3^1(0)\right], & \varepsilon < x_2 < 1. \end{cases}$$

The function $v(\varepsilon)$ thus defined satisfies the following convergence results:

$$\begin{cases} v_i^1(\varepsilon) \to v_i^1 & \text{strongly in } L^2(\Omega^1), \\ e_{11}(v^1)(\varepsilon) \to e_{11}(v^1) & \text{strongly in } L^2(\Omega^1), \\ \varepsilon^{-2} e_{23}(v^1)(\varepsilon) \to \dfrac{1}{2} x_2(1-x_1)(\xi_3^2)'(0) & \text{strongly in } L^2(\Omega^1), \\ \varepsilon^{-1} e_{12}(v^1)(\varepsilon) \to \dfrac{1}{2}\left[\dfrac{1}{2}(\xi_2^1)'(0) + (x_3 - \dfrac{1}{2})(\xi_3^2)'(0)\right] & \text{strongly in } L^2(\Omega^1), \\ \varepsilon^{-1} e_{13}(v^1)(\varepsilon) \to -\dfrac{1}{2} x_2 (\xi_3^2)'(0) & \text{strongly in } L^2(\Omega^1). \end{cases} \tag{41}$$

We prove similar results for $v_2(\varepsilon)$.
Passing to the limit in (20) and using convergences (41), we obtain the result. ■

Remark 2 The following figure illustrates the limiting flexural modes (Fig. 2). □

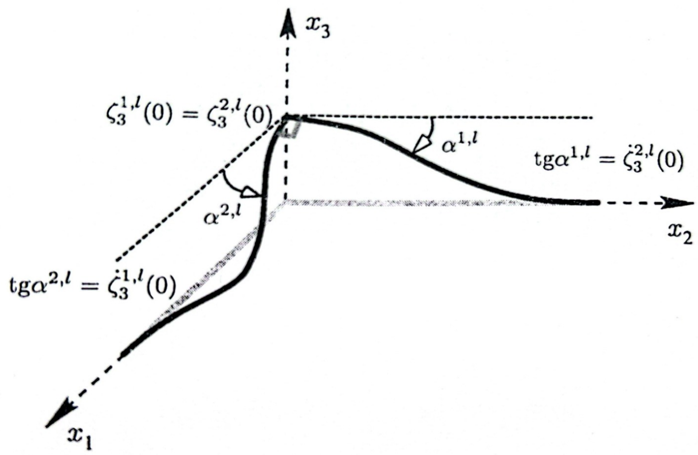

Fig. 2. Limiting flexural modes

Remark 3 The second line of equation (35) highlights the contribution of the torsion experienced by each rod at the junction in the limiting model. This contribution is captured through the torsion angles of the two rods and is explicitly represented by $\left(\zeta_3^{1,\ell}\right)'(0)$ and $\left(\zeta_3^{2,\ell}\right)'(0)$. □

Lemma 3

$$\forall m,n \geq 1, \quad \int_0^1 \zeta_{\alpha^1}^{1,m} \zeta_{\alpha^1}^{1,n} dx_1 + \int_0^1 \zeta_{\alpha^2}^{2,m} \zeta_{\alpha^2}^{2,n} dx_2 = \delta_{mn}. \tag{42}$$

PROOF The result is obtained by passing to the limit in the normalization condition (22). ■

Theorem 2 *The limiting components of the flexion displacements are solutions of the system of classical equations of flexion vibrations:*

$$\begin{cases} -E\dfrac{d^2}{dx_1^2}\left(I^1_{\alpha^1\beta^1}\dfrac{d^2\zeta_{\alpha^1}^{1,\ell}}{dx_1^2}\right) = \mu_\ell(0)\zeta_{\beta^1}^{1,\ell} \quad in \;\; (0,1) \\ -E\dfrac{d^2}{dx_2^2}\left(I^2_{\alpha^2\beta^2}\dfrac{d^2\zeta_{\alpha^2}^{2,\ell}}{dx_2^2}\right) = \mu_\ell(0)\zeta_{\beta^2}^{2,\ell} \quad in \;\; (0,1) \end{cases} \tag{43}$$

with

$$\zeta_{\alpha^1}^{1,\ell}(1) = \zeta_{\alpha^2}^{2,\ell}(1) = (\zeta_{\alpha^1}^{1,\ell})'(1) = (\zeta_{\alpha^2}^{2,\ell})'(1) = 0, \tag{44}$$

$$\zeta_2^{1,\ell}(0) = \zeta_1^{2,\ell}(0) = 0, \tag{45}$$

$$\zeta_3^{1,\ell}(0) = \zeta_3^{2,\ell}(0), \tag{46}$$

$$(\zeta_2^{1,\ell})'(0) = -(\zeta_1^{2,\ell})'(0), \tag{47}$$

□

$$I^1_{\alpha^1 3}\frac{d^3\zeta_{\alpha^1}^{1,\ell}}{dx_1^3}(0) + I^2_{\alpha^2 3}\frac{d^3\zeta_{\alpha^2}^{2,\ell}}{dx_2^3}(0) = 0, \tag{48}$$

$$\begin{cases} I^1_{\alpha^1 2}\dfrac{d^2\zeta_{\alpha^1}^{1,\ell}}{dx_1^2}(0) = I^2_{\alpha^2 2}\dfrac{d^2\zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(0), \\ I^1_{\alpha^1 3}\dfrac{d^2\zeta_{\alpha^1}^{1,\ell}}{dx_1^2}(0) = K^2(\zeta_3^{1,\ell})'(0), \\ I^1_{\alpha^2 3}\dfrac{d^2\zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(0) = K^1(\zeta_3^{2,\ell})'(0). \end{cases} \tag{49}$$

PROOF We obtain the result by performing an integration by parts in the left side of equation (35). ■

4. Convergence of the high frequency modes

The main objective of this paper is to analyze the convergence of high-frequency modes in a multi-rod structure as the rod thickness approaches zero. Unlike the case of low-frequency modes, whose limiting behavior can be characterized using standard methods, the study of high-frequency modes presents significant challenges, requiring a different approach. To address this issue and effectively capture the behavior of these modes, we adopt the strategy introduced in [13] for the case of a thin rod. This approach allows us to reach the high-frequency modes and characterize the corresponding limiting problem in a rigorous manner.

First, let us introduce the new scaled eigenvalues:

$$\eta_m(\varepsilon) = \eta_m^\varepsilon = \varepsilon^2 \mu_m(\varepsilon), \tag{50}$$

and reformulate the scaled variational problem as follows: Find $(\eta_m(\varepsilon), u^m(\varepsilon)) \in \mathbb{R} \times V(\varepsilon)$, such that for all $v(\varepsilon) \in V(\varepsilon)$ the following holds:

$$\begin{aligned}
&\int_{\Omega_1} \bar{b}_\varepsilon^1(u^{1,m}(\varepsilon), v^1(\varepsilon))dx + \int_{\Omega_2 \setminus J_\varepsilon^2} \bar{b}_\varepsilon^2(u^{2,m}(\varepsilon), v^2(\varepsilon))dx \\
&\quad = \eta_m(\varepsilon) \int_{\Omega_1} \left[u_{\alpha^1}^{1,m}(\varepsilon) v_{\alpha^1}^1(\varepsilon) + \varepsilon^2 u_1^{1,m}(\varepsilon) v_1^1(\varepsilon)\right] dx \\
&\quad + \eta_m(\varepsilon) \int_{\Omega_2 \setminus J_\varepsilon^2} \left[u_{\alpha^2}^{2,m}(\varepsilon) v_{\alpha^2}^2(\varepsilon) + \varepsilon^2 u_2^{2,m}(\varepsilon) v_2^2(\varepsilon)\right] dx,
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
\bar{b}_\varepsilon^1(u,v) = {} & \varepsilon^{-2}\Big[2\mu e_{\alpha^1\beta^1}(u) e_{\alpha^1\beta^1}(v) + \lambda e_{\alpha^1\alpha^1}(u) e_{\beta^1\beta^1}(v)\Big] \\
& + \Big[4\mu e_{\alpha^1 1}(u) e_{\alpha^1 1}(v) + \lambda \big(e_{\alpha^1\alpha^1}(u) e_{11}(v) + e_{11}(u) e_{\alpha^1\alpha^1}(v)\big)\Big] \\
& + \varepsilon^2 (\lambda + 2\mu) e_{11}(u) e_{11}(v),
\end{aligned} \tag{52}$$

and the similar expression for $\bar{b}_\varepsilon^2(u,v)$.

From (23) we know that $\eta_m(\varepsilon) \le \delta_m \varepsilon^2$. Consequently, for a fixed index m, letting ε tend to zero implies that the entire sequence $\eta_m(\varepsilon)$ converges to zero.

Let $(\bar{\eta}_m(\varepsilon))_{m\ge 1}$ denote the sequence of torsional and stretching modes of the multi-structure. As established in [17], these modes are arranged as follows:

$$\begin{array}{cccccccccccc}
\eta_1(\varepsilon_1) & \cdots & \bar{\eta}_1(\varepsilon_1) & & \cdots & \bar{\eta}_2(\varepsilon_1) & & \cdots & & & & \\
\eta_1(\varepsilon_2) & \cdots & & \bar{\eta}_1(\varepsilon_2) & & \cdots & \bar{\eta}_2(\varepsilon_2) & \cdots & & & & \\
\vdots & & & & \ddots & & & \ddots & & & & \\
\eta_1(\varepsilon_n) & \cdots & & & & \bar{\eta}_1(\varepsilon_n) & & \cdots & \bar{\eta}_2(\varepsilon_n) & \cdots & & \\
\downarrow & & & & & & \searrow & & & \searrow & & \\
0 & \cdots & & & & & \bar{\eta}_1(0) & \cdots & & \bar{\eta}_2(0) & \cdots &
\end{array}$$

To characterize this class of frequencies, we introduce a subsequence of indices $\{\ell_\varepsilon^m\}_{m\geq 1}$ such that

$$\bar{\eta}_m(\varepsilon) = \eta_{\ell_\varepsilon^m}(\varepsilon)$$

where ℓ_ε^m denotes the position of the m^{th} torsional and stretching modes within the sequence $\{\eta_m(\varepsilon)\}_{m\geq 1}$. The elements of this subsequence are defined by

$$\ell_\varepsilon^m = \max\{j \in \mathbb{N}^* : \eta_j(\varepsilon) \leq \delta_m\}, \tag{53}$$

where δ_m is an increasing sequence of constants defined in (23), satisfying the following properties:

- $\forall \varepsilon > 0, \{\eta_{\ell_\varepsilon^m}(\varepsilon)\}_{m\geq 1}$ is a subsequence of $\{\eta_m(\varepsilon)\}_{m\geq 1}$ that contains the modes associated with the torsional and stretching vibrations of the multi-structure;
- $\forall m \geq 1$, we have $\eta_{\ell_\varepsilon^m}(\varepsilon) \leq \delta_m$;
- The family $\{\ell_\varepsilon^m\}_{\varepsilon>0}$ is increasing, i.e., for $\varepsilon < \varepsilon'$ it holds that $\ell_\varepsilon^m \leq \ell_{\varepsilon'}^m$;
- $\forall \varepsilon > 0$, the sequence $\{\ell_\varepsilon^m\}_{m\geq 1}$ consists of increasing positive integers satisfying $\ell_\varepsilon^m \geq m, \forall m \geq 1$;
- As $\varepsilon \to 0$, we have $\lim_{\varepsilon\to 0} \ell_\varepsilon^m = +\infty$.

Now, to pass to the limit in these sequences and characterize the limiting problem, we begin by establishing the following bounds:

Lemma 4 *For each $m \geq 1$, there exists a constant $C_m > 0$ independent of ε, such that*

$$\begin{aligned} \| u_{\alpha^1}^{1,\ell_\varepsilon^m}(\varepsilon) \|_{L^2[0,1;[H^1(0,1)]^2]} &\leq C_m, \\ \| \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon) \|_{H^1(\Omega_1)} &\leq C_m, \end{aligned} \tag{54}$$

and

$$\begin{aligned} \| u_{\alpha^2}^{2,\ell_\varepsilon^m}(\varepsilon) \|_{L^2[0,1;[H^1(0,1)]^2]} &\leq C_m, \\ \| \varepsilon u_2^{2,\ell_\varepsilon^m}(\varepsilon) \|_{H^1(\Omega_2)} &\leq C_m. \end{aligned} \tag{55}$$

□

PROOF Let us define the scaled strain tensors $\kappa^{1,\ell_\varepsilon^m}(\varepsilon)$ as

$$\begin{aligned} \kappa_{\alpha^1\beta^1}^{1,\ell_\varepsilon^m}(\varepsilon) &= \varepsilon^{-1} e_{\alpha^1\beta^1}(u^{1,\ell_\varepsilon^m}(\varepsilon)), \\ \kappa_{\alpha^1 1}^{1,\ell_\varepsilon^m}(\varepsilon) &= e_{\alpha^1 1}(u^{1,\ell_\varepsilon^m}(\varepsilon)), \\ \kappa_{11}^{1,\ell_\varepsilon^m}(\varepsilon) &= \varepsilon e_{11}(u^{1,\ell_\varepsilon^m}(\varepsilon)), \end{aligned} \tag{56}$$

and the similar formulas for $\kappa^{2,\ell_\varepsilon^m}(\varepsilon)$.

Using the definitions (51) and (52), we have

$$
\begin{aligned}
&2\mu\|\kappa^{1,\ell_\varepsilon^m}(\varepsilon)\|_{L^2(\Omega_1)^9}+2\mu\|\kappa^{2,\ell_\varepsilon^m}(\varepsilon)\|_{L^2(\Omega_2)^9}\\
&\quad\le \int_{\Omega_1}\bar b_\varepsilon\big(u^{1,\ell_\varepsilon^m}(\varepsilon),u^{1,\ell_\varepsilon^m}(\varepsilon)\big)\,dx+\int_{\Omega_2\setminus J_\varepsilon^2}\bar b_\varepsilon\big(u^{2,\ell_\varepsilon^m}(\varepsilon),u^{2,\ell_\varepsilon^m}(\varepsilon)\big)\,dx\\
&\quad=\eta_{\ell_\varepsilon^m}(\varepsilon)\le\delta_m.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
\|\kappa_{ij}^{1,\ell_\varepsilon^m}(\varepsilon)\|_{L^2(\Omega_1)}\le C_m,\\
\|\kappa_{ij}^{2,\ell_\varepsilon^m}(\varepsilon)\|_{L^2(\Omega_2)}\le C_m,
\end{aligned}
\tag{57}
$$

and consequently

$$
\begin{aligned}
\|e_{\alpha^1\beta^1}(u^{1,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_1)} &\le C_m\varepsilon\le C_m,\\
\|e_{\alpha^1 1}(u^{1,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_1)} &\le C_m,\\
\|e_{11}(u^{1,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_1)} &\le C_m\varepsilon^{-1},
\end{aligned}
\tag{58}
$$

and the similar bounds for $e_{ij}(u^{2,\ell_\varepsilon^m}(\varepsilon))$.
Therefore, inequalities (54)-(55) are obtained using Korn inequality in $H^1_{\Gamma_1}(\Omega_1;\mathbb{R}^3)$ and $H^1_{\Gamma_2}(\Omega_2;\mathbb{R}^3)$ respectively. ■

Now, we can pass to the limit in the scaled eigenvalues and eigenvectors.

Lemma 5 *For each $m\ge 1$, there exists a subsequence, still denoted ε such that*

$$
\eta_{\ell_\varepsilon^m}(\varepsilon)\to\bar\eta_m(0),
\tag{59}
$$

$$
\big(u_{\alpha^1}^{1,\ell_\varepsilon^m}(\varepsilon),u_{\alpha^2}^{2,\ell_\varepsilon^m}(\varepsilon)\big)\rightharpoonup\big(\bar u_{\alpha^1}^{1,m},\bar u_{\alpha^2}^{2,m}\big)\quad \textit{weakly in}\ \big(L^2[0,1;[H^1(0,1)]^2]\big)^2
\tag{60}
$$

$$
\big(\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon),\varepsilon u_2^{2,\ell_\varepsilon^m}(\varepsilon)\big)\rightharpoonup\big(\bar u_1^{1,m},\bar u_2^{2,m}\big)\quad \textit{weakly in}\ H^1(\Omega_1)\times H^1(\Omega_2)
\tag{61}
$$

where

$$
\begin{cases}
\bar u_1^{1,m}(x) &=\zeta^{1,m}(x_1),\\
\bar u_2^{1,m}(x) &=(x_3-1/2)\theta^{1,m}(x_1)+\zeta_2^{1,m}(x_1),\\
\bar u_3^{1,m}(x) &=-(x_2-1/2)\theta^{1,m}(x_1)+\zeta_3^{1,m}(x_1),
\end{cases}
\tag{62}
$$

and

$$
\begin{cases}
\bar u_1^{2,m}(x) &=(x_3-1/2)\theta^{2,m}(x_2)+\zeta_1^{2,m}(x_2),\\
\bar u_2^{2,m}(x) &=\zeta^{2,m}(x_2),\\
\bar u_3^{2,m}(x) &=-(x_1-1/2)\theta^{2,m}(x_2)+\zeta_3^{2,m}(x_2),
\end{cases}
\tag{63}
$$

with

$$
\zeta_{\alpha^1}^{1,m},\zeta_{\alpha^2}^{2,m}\zeta^{1,m},\zeta^{2,m},\theta^{1,m},\theta^{2,m}\in H^1([0,1]),
\tag{64}
$$

satisfying the limiting clamping conditions:

$$\zeta^{1,m}_{\alpha^1}(1) = \zeta^{2,m}_{\alpha^2}(1) = \zeta^{1,m}(1) = \zeta^{2,m}(1) = \theta^{1,m}(1) = \theta^{2,m}(1) = 0. \tag{65}$$

PROOF By construction, $\{\eta_{\ell^m_\varepsilon}\}_{m\geq 1}$ is a bounded sequence. Convergences (60)-(61) are deduced from (54)-(55). From (58) we have

$$e_{\alpha^1\beta^1}\big(u^{1,\ell^m_\varepsilon}(\varepsilon)\big) \to 0 \ \text{ strongly in } \ \mathrm{L}^2(\Omega_1), \tag{66}$$

and since

$$u^{1,\ell^m_\varepsilon}_{\alpha^1}(\varepsilon) \rightharpoonup \bar{u}^{1,m}_{\alpha^1} \ \text{ weakly in } \ \mathrm{L}^2[0,1;[\mathrm{H}^1(0,1)]^2], \tag{67}$$

we have

$$e_{\alpha^1\beta^1}\big(u^{1,\ell^m_\varepsilon}(\varepsilon)\big) \rightharpoonup e_{\alpha^1\beta^1}(\bar{u}^{1,m}) \ \text{ weakly in } \ \mathrm{L}^2(\Omega_1). \tag{68}$$

Thus, $e_{\alpha^1\beta^1}(\bar{u}^{1,m}) = 0$. Therefore, we deduce the last two equations in (62). In addition, we deduce from (54) that

$$\varepsilon u^{1,\ell^m_\varepsilon}(\varepsilon) \rightharpoonup (\bar{u}^{1,m}_1, 0, 0) \ \text{ weakly in } \ \mathrm{H}^1(\Omega_1),$$

and

$$e_{i\alpha^1}\big(\varepsilon u^{1,\ell^m_\varepsilon}(\varepsilon)\big) \to 0 \ \text{ strongly in } \ \mathrm{L}^2(\Omega_1). \tag{69}$$

Thus, $\bar{u}^{1,m}_1$ does not depend on x_2 and x_3 which gives the first equation in (62). By the same argument we prove (63). The limiting clamping conditions (65) are directly obtained from the clamping conditions on $\Gamma_1 \cup \Gamma_2$. ■

In order to pass to the limit in the scaled variational formulation, we need the following convergence result regarding the scaled strain tensors.

Lemma 6 *For each $m \geq 1$, there exists a subsequence, still denoted by ε, such that*

$$\kappa^{1,\ell^m_\varepsilon}_{ij}(\varepsilon) \rightharpoonup \kappa^{1,m}_{ij}(0) \quad \text{weakly in } \ \mathrm{L}^2(\Omega_1), \tag{70}$$

$$\kappa^{2,\ell^m_\varepsilon}_{ij}(\varepsilon) \rightharpoonup \kappa^{2,m}_{ij}(0) \quad \text{weakly in } \ \mathrm{L}^2(\Omega_2), \tag{71}$$

with

$$\kappa^{1,m}_{11}(0)(x) = (\zeta^{1,m})'(x_1), \tag{72}$$

$$\kappa^{1,m}_{22}(0)(x) = \kappa^{1,m}_{33}(0)(x) = \frac{-\lambda}{2(\lambda+\mu)}(\zeta^{1,m})'(x_1), \tag{73}$$

$$\kappa^{1,m}_{23}(0)(x) = 0, \tag{74}$$

$$\kappa^{1,m}_{12}(0)(x) = (\theta^{1,m})'(x_1)\partial_3\chi^1(x_2,x_3),$$

$$\kappa^{1,m}_{13}(0)(x) = -(\theta^{1,m})'(x_1)\partial_2\chi^1(x_2,x_3), \tag{75}$$

where χ^1 is the torsional function of ω^1 defined in (38)*, and the similar formulas for $\kappa^{2,m}_{ij}(0)$.* □

PROOF The proof being independent of each rod, we employ the same techniques as in the single rod case (see [13]) to establish this result. ∎

As demonstrated in the following lemma, the flexural components of the limiting displacements disappear.

Lemma 7 *For each* $m \geq 1$*, if* $\bar{\eta}_m(0) \neq 0$ *then*

$$\zeta_{\alpha^1}^{1,m} = \zeta_{\alpha^2}^{2,m} = 0. \tag{76}$$

PROOF Consider a test-function of the form $v(\varepsilon) = (v^1(\varepsilon), 0)$ with

$$v^1(\varepsilon) = (0, \xi_2^1(x_1), \xi_3^1(x_1)), \text{ and } \xi_{\alpha^1}^1 \in \mathscr{D}(0,1). \tag{77}$$

For ε sufficiently small, we have $\xi_{\alpha^1}^1(\varepsilon x_1) = 0$; therefore, this test-function satisfies the junction relations (16) and belongs to $V(\varepsilon)$. Moreover, we have the following strain components:

$$e_{\alpha^1\beta^1}(v^1(\varepsilon)) = 0, \; e_{\alpha^1 1}(v^1(\varepsilon)) = \frac{1}{2}(\xi_{\alpha^1}^1)', \quad \text{and } e_{11}(v^1(\varepsilon)) = 0. \tag{78}$$

Substituting (77) and (78) into equation (51) and taking the limit as $\varepsilon \to 0$, we obtain

$$4\mu \int_{\Omega_1} \kappa_{\alpha^1 1}^{1,m}(0)(\xi_{\alpha^1}^1)' dx = \bar{\eta}_m(0) \int_{\Omega_1} \bar{u}_{\alpha^1}^{1,m} \xi_{\alpha^1}^1 dx. \tag{79}$$

Using (75), we get

$$\int_{\Omega_1} \kappa_{21}^{1,m}(0)(\xi_2^1)' dx = \int_{\omega_1} \partial_3 \chi^1(x_2,x_3) dx_2 dx_3 \int_0^1 (\theta^{1,m})'(\xi_2^1)' dx_1$$

and

$$\int_{\Omega_1} \kappa_{31}^{1,m}(0)(\xi_3^1)' dx = -\int_{\omega_1} \partial_2 \chi^1(x_2,x_3) dx_2 dx_3 \int_0^1 (\theta^{1,m})'(\xi_3^1)' dx_1$$

Since $\int_{\omega_1} \partial_{\alpha^1} \chi^1(x_2,x_3) dx_2 dx_3 = 0$, it follows that $\int_{\Omega_1} \kappa_{\alpha^1 1}^{1,m}(0)(\xi_2^1)' dx = 0$ and then

$$\int_{\Omega_1} \bar{u}_{\alpha^1}^{1,m} \xi_{\alpha^1}^1 dx = 0.$$

Using (62) and the fact that $\int_{\omega_1} (x_{\alpha^1} - \frac{1}{2}) dx_2 dx_3 = 0$, we obtain

$$\int_0^1 \zeta_{\alpha^1}^{1,m} \xi_{\alpha^1}^1 dx_1 = 0 \quad \forall \xi_{\alpha^1}^1 \in \mathscr{D}(0,1).$$

Thus, it follows that

$$\zeta_{\alpha^1}^{1,m} = 0.$$

Applying the same argument with the test function $v(\varepsilon) = (0, v^2(\varepsilon))$ with

$$v^2(\varepsilon) = (\xi_1^2(x_2), 0, \xi_3^2(x_2)), \text{ and } \xi_{\alpha^2}^2 \in \mathscr{D}(0,1), \tag{80}$$

we similarly obtain $\zeta_{\alpha^2}^{2,m} = 0$. ∎

Remark 1 The flexural components vanish in the high-frequency limit, reflecting the well-known fact that torsional and stretching modes in elastic rods occur at higher natural frequencies than flexural ones.

We now derive the limiting junction condition satisfied by the stretching components of the limiting eigenvectors.

Lemma 8 *For each integer $m \geq 1$ we have the following limiting junction conditions:*

$$\zeta^{1,m}(0) = \zeta^{2,m}(0) = 0. \tag{81}$$

PROOF The scaled modes $(u^{1,\ell_\varepsilon^m}(\varepsilon), u^{2,\ell_\varepsilon^m}(\varepsilon))$ satisfy the multidimensional junction relations (16). In order to pass to the limit in these relations, let us define

$$T_1^\varepsilon u_{\alpha^1}^{1,\ell_\varepsilon^m}(\varepsilon)(x_2,x_3) = \frac{1}{\varepsilon}\int_0^\varepsilon u_{\alpha^1}^{1,\ell_\varepsilon^m}(\varepsilon)(x_1,x_2,x_3)dx_1,$$

$$T_1^\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(x_2,x_3) = \frac{1}{\varepsilon}\int_0^\varepsilon \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(x_1,x_2,x_3)dx_2,$$

and the similar relations for $T_2^\varepsilon u_i^{2,\ell_\varepsilon^m}(\varepsilon)(x_1,x_3)$.
Consider the junction relation

$$\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(\varepsilon x_1, x_2, x_3) = u_1^{2,\ell_\varepsilon^m}(\varepsilon)(x_1, \varepsilon x_2, x_3).$$

Integrating both sides, we obtain

$$\int_{\Omega^1} \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(\varepsilon x_1,x_2,x_3)\,dx_1dx_2dx_3 = \int_{\Omega^2} u_1^{2,\ell_\varepsilon^m}(\varepsilon)(x_1,\varepsilon x_2,x_3)\,dx_1dx_2dx_3$$

$$\int_{\omega^1}\frac{1}{\varepsilon}\int_0^\varepsilon \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(x_1,x_2,x_3)\,dx_1dx_2dx_3 = \int_{\omega^2}\frac{1}{\varepsilon}\int_0^\varepsilon u_1^{2,\ell_\varepsilon^m}(\varepsilon)(x_1,x_2,x_3)\,dx_2dx_1dx_3,$$

and then

$$\int_{\omega^1} T_1^\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(x_2,x_3)dx_2dx_3 = \int_{\omega^2} T_2^\varepsilon u_1^{2,\ell_\varepsilon^m}(\varepsilon)(x_1,x_3)dx_1dx_3. \tag{82}$$

Since $u_1^{1,\ell_\varepsilon^m}(\varepsilon) \in H^1(\Omega_1) \hookrightarrow H^1([0,1];L^2(\omega_1))$, $T_1^\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon) \in L^2(\omega_1)$ and we have, for each $m \geq 1$,

$$
\begin{aligned}
&\left\| T_1^\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon) - \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)_{|\omega_1} \right\|_{L^2(\omega_1)}^2 \\
&\quad = \left\| \frac{1}{\varepsilon} \int_0^\varepsilon \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(x_1,x_2,x_3)\, dx_1 - \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)(0,x_2,x_3) \right\|_{L^2(\omega_1)}^2 \\
&\quad \leq C\varepsilon \left\| \varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon) \right\|_{H^1(\Omega_1)}^2 \leq C\varepsilon.
\end{aligned}
$$

Since $\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon)_{|\omega_1} \rightharpoonup \bar{u}_1^{1,m}{}_{|\omega_1}$ in the $H^{\frac{1}{2}}(\omega_1)$ sense, we deduce that

$$
T_1^\varepsilon u_1^{1,\ell_\varepsilon^m}(\varepsilon) \to \bar{u}_1^{1,m}{}_{|\omega_1} \quad \text{strongly in } L^2(\omega_1).
$$

By the same argument, we show that

$$
T_2^\varepsilon u_1^{2,\ell_\varepsilon^m}(\varepsilon) \to \bar{u}_1^{2,m}{}_{|\omega_2} \quad \text{strongly in } L^2(\omega_2).
$$

Now passing to the limit in the relation (82), we obtain

$$
\int_{\omega_1} \bar{u}_1^{1,m}(0,x_2,x_3) dx_2 dx_3 = \int_{\omega_2} \bar{u}_1^{2,m}(x_1,0,x_3) dx_1 dx_3,
$$

which gives

$$
\int_{\omega_1} \zeta^{1,m}(0) dx_2 dx_3 = \int_{\omega_2} \big(x_3 - \frac{1}{2}\big) \theta^{2,m}(0) dx_1 dx_3,
$$

and then $\zeta^{1,m}(0) = 0$. By the same argument we prove that $\zeta^{2,m}(0) = 0$. ■

We can now characterize the limiting space of stretching and torsional displacements:

$$
\begin{aligned}
\mathscr{V}_{ST} = \Big\{ &(\zeta^{1,m}, \theta^{1,m}, \zeta^{2,m}, \theta^{2,m}) \in H^1(0,1;\mathbb{R}^4) : \zeta^{1,m}(0) = \zeta^{2,m}(0) = 0, \\
&\zeta^{1,m}(1) = \zeta^{2,m}(1) = \theta^{1,m}(1) = \theta^{2,m}(1) = 0 \Big\}.
\end{aligned} \tag{83}
$$

Theorem 3 *For each $m \geq 1$, the limits $(\zeta^{1,m}, \theta^{1,m}, \zeta^{2,m}, \theta^{2,m})$ belong to $\mathscr{V}_{ST}$ and satisfy the following equation:*

$\forall (\xi^1, \varphi^1, \xi^2, \varphi^2) \in \mathscr{V}_{ST}$

$$
\begin{aligned}
&EA(\omega^1) \int_0^1 (\zeta^{1,m})'(\xi^1)' dx_1 + 4\mu J^1 \int_0^1 (\theta^{1,m})'(\varphi^1)' dx_1 \\
&\quad + EA(\omega^2) \int_0^1 (\zeta^{2,m})'(\xi^2)' dx_2 + 4\mu J^2 \int_0^1 (\theta^{2,m})'(\varphi^2)' dx_2 \\
&\qquad = \bar{\eta}_m(0) \left[A(\omega^1) \int_0^1 \zeta^{1,m} \xi^1 dx_1 + (I_{22}^1 + I_{33}^1) \int_0^1 \theta^{1,m} \varphi^1 dx_1 \right] \\
&\qquad + \bar{\eta}_m(0) \left[A(\omega^2) \int_0^1 \zeta^{2,m} \xi^2 dx_2 + (I_{11}^2 + I_{33}^2) \int_0^1 \theta^{2,m} \varphi^2 dx_2 \right].
\end{aligned} \tag{84}
$$

where

$$J^1 = \int_{\omega^1} (x_{\alpha^1} - \frac{1}{2}) \partial_{\alpha^1} \chi^1(x_2, x_3) dx_2 dx_3 \tag{85}$$

and

$$J^2 = \int_{\omega^2} (x_{\alpha^2} - \frac{1}{2}) \partial_{\alpha^2} \chi^2(x_1, x_3) dx_1 dx_3 \tag{86}$$

are the torsional rigidity coefficients and $I^1_{\alpha_1\beta_1}, I^2_{\alpha_2\beta_2}$ *the inertia moments of each rod.* □

PROOF Let $(\xi^1, \varphi^1, \xi^2, \varphi^2)$ belong to $\mathscr{V}_{ST}$. We denote by (v^1, v^2) the displacements

$$\begin{cases} v^1(x) = \Big(\varepsilon^{-1}\xi^1(x_1), (x_3 - \frac{1}{2})\varphi^1(x_1), (x_2 - \frac{1}{2})\varphi^1(x_1)\Big) \\ v^2(x) = \Big((x_3 - \frac{1}{2})\varphi^2(x_2), \varepsilon^{-1}\xi^2(x_2), (x_1 - \frac{1}{2})\varphi^2(x_2)\Big). \end{cases} \tag{87}$$

As in [Theorem 1], we construct an approximation $(v^1(\varepsilon), v^2(\varepsilon))$ of (v^1, v^2) that belongs to $V(\varepsilon)$ and satisfies the following convergence properties:

$$\begin{cases} v^1_{\alpha^1}(\varepsilon) \to v^1_{\alpha^1} & \text{strongly in } L^2(\Omega_1), \\ \varepsilon v^1_1(\varepsilon) \to \varphi^1(x_1) & \text{strongly in } L^2(\Omega_1), \\ e_{\alpha^1 1}(v^1(\varepsilon)) \to e_{\alpha^1 1}(v^1) & \text{strongly in } L^2(\Omega_1), \\ \varepsilon e_{11}(v^1(\varepsilon)) \to (\varphi^1)'(x_1) & \text{strongly in } L^2(\Omega_1), \\ \varepsilon^{-1} e_{\alpha^1\beta^1}(v^1(\varepsilon)) \to 0 & \text{strongly in } L^2(\Omega_1). \end{cases} \tag{88}$$

and the similar properties for $v^2(\varepsilon)$.
Passing to the limit in (51) when $\varepsilon \to 0$, and using convergences (70)-(71) and (88), we obtain

$$\begin{aligned} 4\mu \int_{\Omega_1} \kappa^{1,m}_{\alpha^1 1}(0) e_{\alpha^1 1}(v^1) dx &+ \int_{\Omega_1} \Big[2\mu\kappa^{1,m}_{11}(0) + \lambda\kappa^{1,m}_{ii}(0)\Big] (\varphi^1)'(x_1) dx \\ +4\mu \int_{\Omega_2 \setminus J^2_\varepsilon} \kappa^{2,m}_{\alpha^2 2}(0) e_{\alpha^2 2}(v^2) dx &+ \int_{\Omega_2 \setminus J^2_\varepsilon} \Big[2\mu\kappa^{2,m}_{22}(0) + \lambda\kappa^{2,m}_{ii}(0)\Big] (\varphi^2)'(x_2) dx \\ = \bar{\eta}_m(0) \Big[\int_{\Omega_1} \bar{u}^{1,m}_{\alpha^1}(x) v^1_{\alpha^1}(x) dx &+ \int_{\Omega_1} \bar{u}^{1,m}_1(x) \varphi^1(x_1) dx\Big] \\ +\bar{\eta}_m(0) \Big[\int_{\Omega_2 \setminus J^2_\varepsilon} \bar{u}^{2,m}_{\alpha^2}(x) v^2_{\alpha^2}(x) dx &+ \int_{\Omega_2 \setminus J^2_\varepsilon} \bar{u}^{2,m}_2(x) \varphi^2(x_2) dx\Big]. \end{aligned}$$

Replacing $\bar{u}^{1,m}(x)$, $\bar{u}^{2,m}(x)$, $\kappa^{1,m}(0)$, and $\kappa^{2,m}(0)$ by their expressions (62)-(63) and (72)-(75), we obtain

$$
\begin{aligned}
&4\mu \int_{\omega^1} (x_{\alpha^1} - \frac{1}{2})\partial_{\alpha^1}\chi^1(x_2,x_3)dx_2dx_3 \int_0^1 (\theta^{1,m})'(x_1)(\varphi^1)'(x_1)dx_1 \\
&+4\mu \int_{\omega^2} (x_{\alpha^2} - \frac{1}{2})\partial_{\alpha^2}\chi^2(x_1,x_3)dx_1dx_3 \int_0^1 (\theta^{2,m})'(x_2)(\varphi^2)'(x_2)dx_2 \\
&+\frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \int_{\omega^1} dx_2dx_3 \int_0^1 (\zeta^{1,m})'(x_1)(\xi^1)'(x_1)dx_1 \\
&+\frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \int_{\omega^2} dx_1dx_3 \int_0^1 (\zeta^{2,m})'(x_2)(\xi^2)'(x_2)dx_2 \\
&= \bar{\eta}_m(0) \int_{\omega^1} \Big[\big(x_2 - \frac{1}{2}\big)^2 + \big(x_3 - \frac{1}{2}\big)^2\Big] dx_2dx_3 \int_0^1 \theta^{1,m}\varphi^1 dx_1 \\
&+\bar{\eta}_m(0) \int_{\omega^2} \Big[\big(x_1 - \frac{1}{2}\big)^2 + \big(x_3 - \frac{1}{2}\big)^2\Big] dx_1dx_3 \int_0^1 \theta^{2,m}\varphi^2 dx_2 \\
&+\bar{\eta}_m(0) \int_{\omega^1} dx_2dx_3 \int_0^1 \zeta^{1,m}\xi^1 dx_1 + \bar{\eta}_m(0) \int_{\omega^2} dx_1dx_3 \int_0^1 \zeta^{2,m}\xi^2 dx_2.
\end{aligned}
$$

Equation (84) is obtained using definitions (36)-(37), and (85)-(86). ■

Let us now give the strong formulation of the limit variational problem. The lack of strong convergence of the eigenfuctions does not allow us to assure that this limit is not zero. However, if this limit is not zero, then it is a torsional or stretching eigenmode associated to the limit eigenvalue.

Theorem 4 *For each $m \geq 1$, the limiting solution $(\bar{\eta}_m(0), \zeta^m, \theta^m)$ verifies:*
• *If $(\zeta^{1,m}, \zeta^{2,m}) \neq (0,0)$ then $(\bar{\eta}_m(0), \zeta^{1,m}, \zeta^{2,m})$ is an eigensolution of the system of classical equations of stretching vibrations:*

$$
\begin{cases}
-E\,\dfrac{d^2\zeta^1}{dx_1^2} = \eta\zeta^1 \quad \textit{in } (0,1), \\
-E\,\dfrac{d^2\zeta^2}{dx_2^2} = \eta\zeta^2 \quad \textit{in } (0,1).
\end{cases}
\tag{89}
$$

• *If $(\theta^{1,m}, \theta^{2,m}) \neq (0,0)$ then $(\bar{\eta}_m(0), \theta^{1,m}, \theta^{2,m})$ is an eigensolution of the system of classical equations of torsional vibrations:*

$$
\begin{cases}
-4\mu J^1\,\dfrac{d^2\theta^1}{dx_1^2} = \eta(I_{22}^1 + I_{33}^1)\theta^1 \quad \textit{in } (0,1), \\
-4\mu J^2\,\dfrac{d^2\theta^2}{dx_2^2} = \eta(I_{11}^2 + I_{33}^2)\theta^2 \quad \textit{in } (0,1).
\end{cases}
\tag{90}
$$

PROOF The result is obtained by carefully performing an integration by parts in the left side of equation (84). ■

5. Conclusions

In this paper, we performed a detailed asymptotic analysis of the eigenvalue problem for a multi-rod structure clamped at both ends within the framework of linear elasticity. We investigated the convergence of both low-frequency and high-frequency modes as the thickness of the rods tends to zero. For low-frequency modes, we established the convergence of eigenvalues and eigenfunctions to a well-posed one--dimensional limiting problem. Specifically, we showed that:

- The eigenfunctions primarily correspond to flexural displacements in each rod. The limiting problem is governed by coupled fourth-order differential equations with appropriate junction conditions.
- The derived junction conditions ensure the continuity of displacements and maintain the perpendicularity of the rod axes after deformation.
- These results confirm that the limiting model accurately describes the behavior of the multi-rod structure under low-frequency vibrations.

The study of high-frequency modes is more complicated, and the classical techniques used for low-frequency modes could not be applied in this setting. To address this, we:

- Defined a subsequence of eigenvalues associated with torsional and stretching modes and characterized their asymptotic behavior.
- Established the convergence properties of the scaled eigenfunctions.
- Derived the governing equations for the high-frequency limiting problem, demonstrating that these modes correspond to pure torsional and stretching vibrations.

This work specifically concerns a multi-rod structure that is clamped at both ends. The case of a multi-rod structure clamped at only one end for high-frequency modes presents additional complexities and will be addressed in a forthcoming study. Our findings provide a rigorous mathematical justification for reduced-dimensional models in multi-rod structures, capturing both flexural (low-frequency) and torsional/stretching (high-frequency) behaviors. Future work may explore numerical simulations to validate these asymptotic results and extend the analysis to more complex multi-rod configurations with varying boundary conditions.

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