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OPERATORS WITH MEMORY IN SCHRAMM SPACES

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Abstract. We show that every operator with memory acting between Banach spaces $C\Phi BV(I)$ of continuous functions of bounded variation in the sense of Schramm defined on a compact interval I of a real axis, is a Nemytskij composition operator with the continuous generating function. Moreover, some consequences for uniformly bounded operators with memory will be given. As a by-product, we obtain that a Banach space $C\Phi BV(I)$ has the uniform Matkowski property.

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1. Introduction

An operator K acting between two classes of functions $f : I \to \mathbb{R}$, where I is a compact real interval, is called *an operator with memory* (or *locally defined*) if whenever two functions f and g from the first class coincide on some open interval $J \subset I$, their images $K(f)$ and $K(g)$ in the second class also coincide on J. A typical example is the Nemytskij composition operator

$$H(f)(x) = h(x, f(x)) \quad (x \in I),$$

generated by some function $h : I \times \mathbb{R} \to \mathbb{R}$. It turns out that in the special cases the inverse is also true, i.e., every operator with memory has to be a Nemytskij composition operator with suitable h. However, the property of "memory" is not sufficient for characterizing all such operators; it largely depends on the function spaces which are its domain and range. For example, Lichawski et al. [1] proved that every operator with memory mapping the space $C^m(I)$ of m-times continuously differentiable functions on I into the space $C(I)$ of continuous functions defined on the same interval has to be of the form

$$K(f)(x) = h\left(x, f(x), f'(x), \ldots, f^{(m)}(x)\right), \quad f \in C^m(I), \; x \in I,$$

for some function $h : I \times \mathbb{R}^{m+1} \to \mathbb{R}$. There is still an open question about the form of operators with memory of the type $K : C^m(I) \to C^k(I)$, where $k \geq 2$; only a partial solution was given in [2]. For $k = 1$ the problem was solved in [1].

In the present paper, in Section 3, we give a representation formula for operators with memory which are self-mappings of Banach spaces $C\Phi BV(I)$ of continuous functions of bounded variation in the sense of Schramm (Theorem 1). Namely, we show that they are Nemytskij composition operators with the continuous generating function h. In Section 4, under the additional assumption that the operators with memory are uniformly bounded, we observe that operators of such a type must be affine (Theorem 3), i.e., must be of the form

$$K(f) = \alpha \cdot f + \beta,$$

where α and β are the elements from the range. This, among others, generalizes the earlier results obtained for local operators mapping the Banach spaces of continuous functions of bounded variation in the sense of Wiener [3] or in the sense of Waterman [4]. Besides, this gives a representation formula for operators with memory acting between Schramm-Waterman spaces [5] have not been examined so far. Moreover, as a by-product, we get that a Banach space $C\Phi BV(I)$ has the uniform Matkowski property.

The theory of Nemytskij composition operator is closely connected with the study of solutions of nonlinear integral equations of the Hammerstein type and the Volterra--Hammerstein type. The wide spectrum of applications of Volterra integral equations can be found, for instance, in the monograph [6]. Those applications are related to the mathematical modelling of phenomena (physical, biological, and other) where the memory effects play a key role. The smoothness properties of Nemytskij superposition operators are used, for instance, in traveling wave models describing nonlinear dynamics in semiconductor lasers [7], or in material modelling [8] and in economy for certain problems of option pricing within the Black-Scholes model for time-dependent volatility [9].

2. Preliminaries

Recall that a function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is called a *Young function* ([10], Definition 0.16) if φ is continuous, convex, and satisfies $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$, and $\lim_{t \longrightarrow \infty} \varphi(t) = \infty$. Note that ([11], Remark 2.1, also [10], Lemma 1.36) every Young function φ is increasing, superadditive, that is, $\varphi(t_1) + \varphi(t_2) \leq \varphi(t_1 + t_2)$, $t_1 \geq 0$, $t_2 \geq 0$, and, by the convexity,

$$\varphi(\lambda t) \leq \lambda \varphi(t), \quad t \geq 0, \qquad \lambda \in [0,1]. \tag{1}$$

A sequence $\Phi = (\varphi_i)_{i=1}^{\infty}$ of Young functions is said to be a *Schramm sequence* if $\sum_{i=1}^{\infty} \varphi_i(t) = \infty$ for all $t > 0$ and

$$\varphi_{i+1}(t) \leq \varphi_i(t), \quad t > 0,\ i \in \mathbb{N}. \tag{2}$$

Further, let $[a,b]$ be a compact interval. Denote by $\mathscr{S}([a,b])$ the family of all finite non-ordered collections $S_j = \{[a_1,b_1], [a_2,b_2], ..., [a_j,b_j]\}$, $j \in \mathbb{N}$, of non-overlapping intervals $[a_i,b_i] \subset [a,b]$, $i = 1,...,j$, i.e., $(a_i,b_i) \cap (a_l,b_l) = \emptyset$, $i,l \in \{1,...,j\}$, $i \neq l$.

Given an interval $I = [a,b]$, a function $f : [a,b] \to \mathbb{R}$, a collection $S_j \in \mathscr{S}([a,b])$ and a Schramm sequence $\Phi = (\varphi_i)_{i=1}^{\infty}$, we set

$$var_{\Phi}(f,S_j) := \sum_{i=1}^{j} \varphi_i(|f(b_i) - f(a_i)|), \qquad j \in \mathbb{N}. \tag{3}$$

We say that a function f is of *bounded Φ-variation in the sense of Schramm in I (or bounded Schramm variation)*, if the *Φ-variation of f on I*, defined by

$$Var_{\Phi}(f) = Var_{\Phi}(f,I) := \sup\{var_{\Phi}(f,S_j) : S_j \in \mathscr{S}([a,b]),\ j \in \mathbb{N}\}, \tag{4}$$

is finite.

Here and subsequently, by $(\Phi BV(I), \|\cdot\|_{\Phi})$ we denote a Banach space ([10], Proposition 2.44) of all real functions f defined on an interval $I = [a,b]$ such that $Var_{\Phi}\left(\frac{f}{\lambda}\right) < \infty$ for some constant $\lambda > 0$ with the norm

$$\|f\|_{\Phi} := |f(a)| + p_{\Phi}(f), \quad f \in \Phi BV(I), \tag{5}$$

where the Luxemburg-Nakano-Orlicz seminorm p_{Φ} is defined as

$$p_{\Phi}(f) = p_{\Phi}(f,I) := \inf\left\{\varepsilon > 0 : Var_{\Phi}\left(\frac{f}{\varepsilon}\right) \leq 1\right\},\ \ f \in \Phi BV(I).$$

Note that for $\varphi_i(u) = u$, $i \in \mathbb{N}$, the condition (4) coincides with the classical concept of variation in the sense of Jordan; for $\varphi_i(u) = u^p$, $p > 1$, $i \in \mathbb{N}$, in the sense of Wiener; for $\varphi_i(u) = \lambda_i u$, $i \in \mathbb{N}$, where a nonincreasing sequence of positive reals $(\lambda_i)_{i\in\mathbb{N}}$ is a Waterman sequence (i.e., if $\lambda_i \to 0$ as $i \to \infty$ and $\sum_{i=1}^{\infty} \lambda_i = \infty$) in the sense of Waterman; for $\varphi_i(u) = \lambda_i \varphi(u)$, $i \in \mathbb{N}$, where $(\lambda_i)_{i\in\mathbb{N}}$ is a Waterman sequence and φ is a Young function, in the Schramm-Waterman sense. For more information, the reader is referred to [10].

Now, let us quote two lemmas concerning the properties of the functional Var_{Φ} which will be useful.

Lemma 1 ([12], Lemma 1). Given a Schramm sequence $\Phi = (\varphi_i)_{i=1}^{\infty}$, the functional Var_{Φ} is:

(i) nondecreasing (with respect to the intervals), that is if I_1, I_2 are sub-intervals of I and $I_1 \subset I_2$, then

$$Var_\Phi(f, I_1) \leq Var_\Phi(f, I_2);$$

(ii) superadditive (with respect to the intervals), that is if I_1, I_2 are sub-intervals of I such that $I_1 \cap I_2$ is a singleton, then

$$Var_\Phi(f, I_1) + Var_\Phi(f, I_2) \leq Var_\Phi(f, I_1 \cup I_2);$$

(iii) sequentially lower semicontinuous, that is

$$Var_\Phi(f, I) \leq \liminf_{n\to\infty} Var_\Phi(f_n, I),$$

if $f_n \in \mathbb{R}^I$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in I$.

Lemma 2 ([12], Lemma 3). *Let $I = [a,b] \subset \mathbb{R}$ $(a, b \in \mathbb{R},\ a < b)$, $n \in \mathbb{N}$, $(x_i, y_i) \in I \times \mathbb{R}$, $i = 0, \ldots, n$, such that*

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

be fixed and let $\Phi = (\varphi_i)_{i=1}^\infty$ be a Schramm sequence. Then the continuous function $f : I \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} y_i \quad \text{if} \quad t = x_i, \quad i = 0, \ldots, n, \\ \text{affine otherwise} \end{cases}$$

is of bounded Schramm variation on I and

$$Var_\Phi(f) = \max\left\{ \sum_{i=1}^{s} \varphi_{\sigma_s(i)}\left(\left|y_{k_i} - y_{k_{i-1}}\right|\right) : \sigma_s \in \Delta_s,\ s \in \{1, ..., n\},\ y_{k_0} = y_0,\ y_{k_s} = y_n \right\};$$

here Δ_s stands for the set of all permutations of the set $\{1, ..., s\}$.

We will need also the following relationship between two Schramm sequences Φ and Ψ under which one of the corresponding Schramm spaces is contained in the other.

Given two Schramm sequences $\Phi = (\varphi_i)_{i=1}^\infty$ and $\Psi = (\psi_i)_{i=1}^\infty$, by ([10], Proposition 2.45), we have $\Phi BV(I) \subseteq \Psi BV(I)$ if

$$\sum_{i=1}^{n} \varphi_i(t) \geq C \sum_{i=1}^{n} \psi_i(t), \quad t \in [0, T], \quad n \in \mathbb{N}, \tag{6}$$

for some $T > 0$ and $C > 0$.

3. The form of operators with memory in Schramm spaces

It is known ([10], Proposition 2.46) that functions of bounded Schramm variation have discontinuities of the first kind and, therefore, the discontinuity set is at most countable. Moreover, the convergence with respect to the norm $\|\cdot\|_\Phi$ implies the uniform convergence on I [[10], Proposition 2.44], so $(\Phi BV(I)\cap C(I), \|\cdot\|_\Phi)$ is a closed subspace of $(\Phi BV(I), \|\cdot\|_\Phi)$, and, consequently, it is a Banach space. Put $C\Phi BV(I) := \Phi BV(I)\cap C(I)$.

To give the main result of this section, recall the following definition.

Definition 1. Let $I=[a,b]\subset\mathbb{R}$ be an interval and let $X(I)$ and $Y(I)$ stand for two classes of functions $f: I\to\mathbb{R}$.

An operator $K: X(I)\to Y(I)$ is said to be an *operator with memory (or locally defined)*, if for every open interval $J\subset\mathbb{R}$ and for all functions $f,g\in X(I)$ the following implication holds true:

$$f|_{J\cap I} = g|_{J\cap I} \Rightarrow K(f)|_{J\cap I} = K(g)|_{J\cap I}.$$

An operator $H: X(I)\to Y(I)$ given by

$$H(f)(x) := h(x, f(x)), \quad f\in X(I),\ (x\in I),$$

for some function $h: I\times\mathbb{R}\to\mathbb{R}$ is said to be a *composition (Nemytskij or superposition) operator*. The function h is referred to as the *generator of the operator H*.

The representation theorem for operators with memory acting between the Banach spaces of continuous functions of bounded Schramm variations reads as follows:

Theorem 1. *Let $\Phi=(\varphi_i)_{i=1}^\infty$ and $\Psi=(\psi_i)_{i=1}^\infty$ be two Schramm sequences and $I=[a,b]$, $a,b\in\mathbb{R}$, $a<b$, be a compact interval. If an operator with memory K maps $C\Phi BV(I)$ into $C\Psi BV(I)$, then it is a Nemytskij composition operator with continuous generator, i.e., there exists a unique continuous function $h: I\times\mathbb{R}\to\mathbb{R}$ such that, for all $f\in C\Phi BV(I)$,*

$$K(f)(x) = h(x, f(x)), \quad x\in I. \tag{7}$$

Proof. We start by showing that the following implication holds true:

$$f(x_0) = g(x_0) \Longrightarrow K(f)(x_0) = K(g)(x_0), \tag{8}$$

for all $f,g\in C\Phi BV(I)$ and $x_0\in I$.

Fix $x_0\in \mathrm{int}I$, two functions $f,g\in C\Phi BV(I)$ coinciding at x_0 and define $\gamma: [a,b]\to\mathbb{R}$ by

$$\gamma(x) = \begin{cases} f(x) & \text{for} \quad x\in[a,x_0] \\ g(x) & \text{for} \quad x\in(x_0,b] \end{cases}.$$

To prove that $\gamma \in C\Phi BV(I)$, take any collection $S_j = \{[a_1,b_1], [a_2,b_2], ..., [a_j,b_j]\}$ pairwise non-overlapping sub-intervals of $[a,b]$ such that $a_k \le x_0 < b_k$ for some $k \in \{1,...,j\}$. Since f and g have bounded Schramm variation, $Var_\Phi\left(\frac{f}{\lambda}\right) < \infty$ and $Var_\Phi\left(\frac{g}{\mu}\right) < \infty$ for some $\lambda, \mu > 0$. Being monotonically increasing on $[a,b]$, functions φ_i fulfill the inequalities

$$\varphi_i\left(\left|\frac{f}{\lambda+\mu}(b_i) - \frac{f}{\lambda+\mu}(a_i)\right|\right) \le \varphi_i\left(\left|\frac{f}{\lambda}(b_i) - \frac{f}{\lambda}(a_i)\right|\right), \quad i \in \{1,...,k-1\},$$

and

$$\varphi_i\left(\left|\frac{g}{\lambda+\mu}(b_i) - \frac{g}{\lambda+\mu}(a_i)\right|\right) \le \varphi_i\left(\left|\frac{g}{\mu}(b_i) - \frac{g}{\mu}(a_i)\right|\right), \quad i \in \{k+1,...,j\}.$$

Furthermore, the equality $f(x_0) = g(x_0)$, monotonicity and convexity of φ_k, gives

$$\begin{aligned}
\varphi_k\left(\left|\frac{\gamma}{\lambda+\mu}(b_k) - \frac{\gamma}{\lambda+\mu}(a_k)\right|\right) &= \varphi_k\left(\left|\frac{g}{\lambda+\mu}(b_k) - \frac{f}{\lambda+\mu}(a_k)\right|\right) \\
&\le \varphi_k\left(\frac{\mu}{\lambda+\mu}\cdot\frac{|g(b_k)-g(x_0)|}{\mu} + \frac{\lambda}{\lambda+\mu}\cdot\frac{|f(x_0)-f(a_k)|}{\lambda}\right) \\
&\le \frac{\mu}{\lambda+\mu}\varphi_k\left(\frac{|g(b_k)-g(x_0)|}{\mu}\right) + \frac{\lambda}{\lambda+\mu}\varphi_k\left(\frac{|f(x_0)-f(a_k)|}{\lambda}\right) \\
&\le \varphi_k\left(\frac{|f(x_0)-f(a_k)|}{\lambda}\right) + \varphi_k\left(\frac{|g(b_k)-g(x_0)|}{\mu}\right).
\end{aligned}$$

Combining these estimates yields

$$\begin{aligned}
var_\Phi(\frac{\gamma}{\lambda+\mu}, S_j) &= \sum_{i=1}^{k-1}\varphi_i\left(\left|\frac{f}{\lambda+\mu}(b_i) - \frac{f}{\lambda+\mu}(a_i)\right|\right) + \varphi_k\left(\left|\frac{g}{\lambda+\mu}(b_k) - \frac{f}{\lambda+\mu}(a_k)\right|\right) \\
&\quad + \sum_{i=k+1}^{j}\varphi_i\left(\left|\frac{g}{\lambda+\mu}(b_i) - \frac{g}{\lambda+\mu}(a_i)\right|\right) \\
&\le \left(\sum_{i=1}^{k-1}\varphi_i\left(\left|\frac{f}{\lambda}(b_i) - \frac{f}{\lambda}(a_i)\right|\right) + \varphi_k\left(\left|\frac{f}{\lambda}(x_0) - \frac{f}{\lambda}(a_k)\right|\right)\right) \\
&\quad + \left(\varphi_k\left(\left|\frac{g}{\mu}(b_k) - \frac{g}{\mu}(x_0)\right|\right) + \sum_{i=k+1}^{j}\varphi_i\left(\left|\frac{g}{\mu}(b_i) - \frac{g}{\mu}(a_i)\right|\right)\right),
\end{aligned}$$

whence, by Lemma 1(i),

$$var_\Phi(\frac{\gamma}{\lambda+\mu}, S_l) \le Var_\Phi\left(\frac{f}{\lambda}\right) + Var_\Phi\left(\frac{g}{\mu}\right) < \infty.$$

Thus, (3) and (4) imply that $Var_\Phi\left(\dfrac{\gamma}{\lambda+\mu}\right) < \infty$, which proves that $\gamma \in C\Phi BV(I)$, by continuity of γ.

Since

$$f|_{(-\infty,x_0)\cap I} = \gamma|_{(-\infty,x_0)\cap I}, \quad g|_{(x_0,\infty)\cap I} = \gamma|_{(x_0,\infty)\cap I},$$

by Definition 1, we get

$$K(f)|_{(-\infty,x_0)\cap I} = K(\gamma)|_{(-\infty,x_0)\cap I}, \ \ K(g)|_{(x_0,\infty)\cap I} = K(\gamma)|_{(x_0,\infty)\cap I}.$$

Therefore, by the continuity of $K(f)$, $K(\gamma)$ and $K(g)$ at x_0, we obtain

$$K(f)(x_0) = K(\gamma)(x_0) = K(g)(x_0).$$

To show that (8) holds true at the endpoints of the interval I, suppose that $x_0 = a$ and take any $f, g \in C\Phi BV(I)$ such that

$$f(x_0) = g(x_0). \tag{9}$$

Taking into account the continuity of f and g at x_0, we get the existence of a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$x_0 < x_{n+1} < x_n, \quad |x_n - x_0| < \frac{b-x_0}{n}, \quad n \in \mathbb{N},$$

and

$$|f(x_n) - f(x_0)| < \frac{1}{n^2}, \quad |g(x_n) - g(x_0)| < \frac{1}{n^2}, \quad n \in \mathbb{N}. \tag{10}$$

Define the sequence of continuous functions $\gamma_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, by

$$\gamma_{2m}(t) = \begin{cases} f(t) & \text{for} \quad t = x_0 \quad \text{or} \quad t = x_{2i},\ i \in \{1,\dots,m\} \\ g(t) & \text{for} \quad t = b \quad \text{or} \quad t = x_{2i-1},\ i \in \{1,\dots,m\} \\ \text{affine otherwise} & \end{cases},$$

$$\gamma_{2m-1}(t) = \begin{cases} g(t) & \text{for} \quad t = b \quad \text{or} \quad t = x_{2i-1},\ i \in \{1,\dots,m\} \\ f(t) & \text{for} \quad t = x_0 \quad \text{or} \quad t = x_{2i-2},\ i \in \{2,\dots,m\} \\ \text{affine otherwise} & \end{cases},$$

for all $m \in \mathbb{N}$. We claim that $\gamma_n \in \Phi BV(I)$, $n \in \mathbb{N}$. In fact, using Lemma 2 with $a = x_0 < x_{2m} < x_{2m-1} < \ldots < x_1 < b$ and the definition of γ_{2m}, we get, for any $m \in \mathbb{N}$,

$$Var_\Phi(\gamma_{2m}) = \max\left\{\sum_{i=1}^{s} \varphi_{\sigma_s(i)}\left(\left|\gamma_{2m}(x_{k_i}) - \gamma_{2m}\left(x_{k_{i-1}}\right)\right|\right) : \sigma_s \in \Delta_s\,,\ s \in \{1,...,2m\}\right\}, \tag{11}$$

where $x_{k_0} = x_0,\ x_{k_s} = b,\ s \in \{1,...,2m\}$.

Taking an arbitrary $m \in \mathbb{N}$, $s \in \{1,...,2m\}$ and $\sigma_s \in \Delta_s$, by the definition of γ_{2m}, $m \in \mathbb{N}$, the triangle inequality, (9) and (10), we have

$$|\gamma_{2m}(x_{k_1}) - \gamma_{2m}(x_0)| \le \frac{1}{k_1^2}$$

and

$$\begin{aligned} & |\gamma_{2m}(x_{k_i}) - \gamma_{2m}\left(x_{k_{i-1}}\right)| \\ \le\ & |\gamma_{2m}(x_{k_i}) - \gamma_{2m}(x_0)| + |\gamma_{2m}\left(x_{k_{i-1}}\right) - \gamma_{2m}(x_0)| \le \frac{1}{k_i^2} + \frac{1}{k_{i-1}^2} < \frac{2}{k_{i-1}^2}, \end{aligned}$$

for all $i \in \{2,\dots,s\}$. Hence, applying the monotonicity of φ_1, inequality (1) with $\lambda = \dfrac{1}{k_{i-1}^2}$, $i \in \{2,\dots,s\}$, we get

$$\varphi_1(|\gamma_{2m}(x_{k_1}) - \gamma_{2m}(x_0)|) \le \varphi_1(\frac{1}{k_1^2}) \le \frac{1}{k_1^2}\varphi_1(1) \le \frac{1}{k_1^2}\varphi_1(2), \tag{12}$$

and

$$\varphi_1(|\gamma_{2m}(x_{k_i}) - \gamma_{2m}\left(x_{k_{i-1}}\right)|) \le \varphi_1(\frac{2}{k_{i-1}^2}) \le \frac{1}{k_{i-1}^2}\varphi_1(2), \quad i \in \{2,\dots,s\}. \tag{13}$$

Therefore, by (2), (11), (12) and (13), we obtain

$$Var_\Phi(\gamma_{2m}) \le \max\left\{\frac{1}{k_1^2}\varphi_1(2) + \sum_{i=1}^{s-1}\frac{1}{k_i^2}\varphi_1(2) :\ \sigma_s \in \Delta_s\,,\ s \in \{2,...,2m\}\right\},$$

whence

$$Var_\Phi(\gamma_{2m}) \le 2\varphi_1(2)\sum_{i=1}^{2m}\frac{1}{i^2}, \quad m \in \mathbb{N}. \tag{14}$$

Similar reasoning shows that

$$Var_\Phi(\gamma_{2m-1}, I) \le 2\varphi_1(2)\sum_{i=1}^{2m-1}\frac{1}{i^2}, \quad m \in \mathbb{N},$$

which, together with (14), implies that $\gamma_n \in \Phi BV(I)$ and

$$Var_\Phi(\gamma_n, I) \le 2\varphi_1(2)\sum_{i=1}^{n}\frac{1}{i^2}, \quad n \in \mathbb{N}. \tag{15}$$

Let us note that, by the definition of γ_n,

$$\gamma_n(x_0) = f(x_0) = g(x_0), \quad n \in \mathbb{N}, \tag{16}$$

and, for all $m \in \mathbb{N}$, $i \in \mathbb{N}$,

$$\gamma_{2m}(x_{2m}) = f(x_{2m}) = \gamma_{2m+i}(x_{2m}), \quad \gamma_{2m-1}(x_{2m-1}) = g(x_{2m-1}) = \gamma_{2m-1+i}(x_{2m-1}). \tag{17}$$

Moreover, let us observe that for every $x \in I \backslash \{x_n : n \in \mathbb{N}\}$ there exists $n_0 \in \mathbb{N}$ such that

$$\gamma_n(x) = \gamma_{n_0}(x), \quad n \geq n_0,\ n \in \mathbb{N}. \tag{18}$$

Put

$$\gamma(t) := \lim_{n\to\infty} \gamma_n(t), \quad t \in I.$$

By (16), (17) and (18) the function γ is well defined. Moreover, by Lemma 1(*iii*) and (15),

$$Var_\Phi(\gamma, I) \leq \liminf_{n\to\infty} \left(2\varphi_1(2) \sum_{i=1}^{n} \frac{1}{i^2} \right),$$

therefore the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$ implies that $\gamma \in \Phi BV(I)$. Since, by the continuity of f and g and

$$\sup\{|\gamma_n(x) - \gamma(x)| : x \in [a,b]\} \leq \sup\{|f(x) - g(x)| : x \in [a, x_n]\},$$

the sequence $(\gamma_n)_{n\in\mathbb{N}}$ tends uniformly to γ, the function γ is continuous.

Thus there exists a function $\gamma \in C\Phi BV(I)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\gamma(x_{2n-1}) = g(x_{2n-1}), \quad \gamma(x_{2n}) = f(x_{2n}), \quad x_n \in I,\ n \in \mathbb{N}.$$

According to the first part of the proof, we have

$$K(\gamma)(x_{2n-1}) = K(g)(x_{2n-1}), \quad K(\gamma)(x_{2n}) = K(f)(x_{2n}),\ n \in \mathbb{N}.$$

Hence, by the continuity of $K(\gamma)$, $K(f)$ and $K(g)$ at x_0, letting $n \to \infty$, we get (8).

When x_0 is the right endpoint of I, the argumentation is similar.

Now, we are in a position to construct the function h. For an arbitrary $y_0 \in \mathbb{R}$ let us define a function $P_{y_0} : I \to \mathbb{R}$ by

$$P_{y_0}(t) := y_0, \quad t \in I. \tag{19}$$

Of course P_{y_0}, as a constant function, belongs to $C\Phi BV(I)$. To define the function $h : I \times \mathbb{R} \to \mathbb{R}$, fix arbitrarily $x_0 \in I$, $y_0 \in \mathbb{R}$ and put

$$h(x_0, y_0) := K(P_{y_0})(x_0).$$

Since, by (19), for all functions f,

$$f(x_0) = P_{f(x_0)}(x_0),$$

according to what has already been proved, we have

$$K(f)(x_0) = K(P_{f(x_0)})(x_0) = h(x_0, f(x_0))$$

which means that K is a Nemytskij composition operator with a generator h. We obtain the continuity of the function h by Theorem 1 of [12]. As the uniqueness of the function h is obvious, the proof is completed. □

Remark 1. Taking $\varphi_i(u) = \varphi_1(u)$, $i \in \mathbb{N}$, one gets the main result of [3] and taking $\varphi_i(u) = \lambda_i u$, $i \in \mathbb{N}$, where $(\lambda_i)_{i\in\mathbb{N}}$ is a Waterman sequence (i.e., if $\lambda_i \to 0$ as $i \to \infty$ and $\sum_{i=1}^{\infty} \lambda_i = \infty$) the main result of [4].

4. Uniformly bounded operators with memory in the spaces of continuous functions of bounded Φ-variation

Let us start by recalling some basic facts concerning the uniformly bounded Nemytskij composition operators acting between Schramm spaces.

Definition 2 ([13], Definition 4.1). Let $\mathcal{Y}$ and $\mathcal{Z}$ be metric (or normed) spaces. A mapping $H : \mathcal{Y} \to \mathcal{Z}$ is said to be *uniformly bounded*, if for any $t > 0$ there is a nonnegative real number $\gamma(t)$ such that for any set $B \subset \mathcal{Y}$ we have

$$\operatorname{diam} B \le t \Rightarrow \operatorname{diam} H(B) \le \gamma(t).$$

Definition 3 ([10], Definition 6.26). We say that a pair $(X(I), Y(I))$ of normed function spaces $(X(I), \|\cdot\|_X)$ and $(Y(I), \|\cdot\|_Y)$ has *the uniform weak Matkowski property,* if whenever the Nemytskij composition operator H maps the space $(X(I), \|\cdot\|_X)$ into the space $(Y(I), \|\cdot\|_Y)$ and is uniformly bounded, the corresponding generating *left regularization of* h, i.e., the function $h^- : I^- \times \mathbb{R} \to \mathbb{R}$ defined by

$$h^-(x,y) := \lim_{s\uparrow x} h(s,y), \quad x \in I^- = (a,b]; \quad y \in \mathbb{R},$$

must have the form

$$h^-(x,y) = \alpha(x)y + \beta(x), \quad x \in (a,b],\ y \in \mathbb{R},$$

for some functions $\alpha, \beta \in Y(I)$.

Similarly, we say that $(X(I), Y(I))$ has *the uniform Matkowski property,* if the generator h of the uniformly bounded Nemytskij superposition operator $H : X(I) \to Y(I)$ has the form

$$h(x,y) = \alpha(x)y + \beta(x), \quad x \in [a,b],\ y \in \mathbb{R},$$

for some functions $\alpha,\beta \in Y(I)$.

Theorem 2 ([10], Theorem 6.29). *Let $X(I)$ and $Y(I)$ be two function spaces over $[a,b]$ such that the space $P_n([a,b])$ of polynomials of degree not exceeding n, equipped with the norm of $X(I)$, is imbedded into $X(I)$ for each $n \in N$ and $Y(I)$ is imbedded into some space $\Phi BV([a,b])$ of functions of bounded Schramm variation with norm (5). If a generator $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ of a corresponding Nemytskij composition operator acting between $X(I)$ and $Y(I)$ is such that for any $x\in[a,b]$ a function $h(x,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous with respect to the second variable, then $(X(I),Y(I))$ has the uniform weak Matkowski property.*

Remark 2. In the original version of Theorem 2, the assumption that for any $x\in[a,b]$ a function $h(x,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous with respect to the second variable is omitted, but it is necessary for the correctness of the proof.

Let us recall that a function space $(X(I),\|\cdot\|_X)$ is *imbedded* into another function space $(Y(I),\|\cdot\|_Y)$ (denoted by $X\hookrightarrow Y$), if $X(I)\subseteq Y(I)$ and

$$\|f\|_Y \le c\,\|f\|_X\,, \qquad f\in X(I)\,,$$

for some *imbedding constant* $c>0$ independent of f.

Now, we are in a position to give a characterization formula for operators with memory in the spaces of continuous functions of bounded Φ-variation which are, additionally, uniformly bounded.

Theorem 3. Let $\Phi=(\varphi_i)_{i=1}^{\infty}$ and $\Psi=(\psi_i)_{i=1}^{\infty}$ *be two Schramm sequences and I be a compact interval of a real axis. If an operator with memory K mapping $C\Phi BV(I)$ into $C\Psi BV(I)$ is uniformly bounded, then there exist $\alpha(\cdot)$ and $\beta(\cdot)\in C\Psi BV(I)$ such that*

$$K(f)(x)=\alpha(x)f(x)+\beta(x), \quad f\in C\Phi BV(I), \quad (x\in I). \tag{20}$$

Conversely, if there exist $T>0$ and $C>0$ such that (6) is fulfilled, and an operator $K:\mathbb{R}^I\to\mathbb{R}^I$ is defined by (20) for some functions $\alpha,\beta\in C\Psi BV(I)$, then the operator K maps $C\Phi BV(I)$ into $C\Psi BV(I)$, has the memory property and is uniformly bounded.

Proof. By Theorem 1, every operator with memory K mapping $C\Phi BV(I)$ into $C\Psi BV(I)$ is the Nemytskij composition operator, that is of the form (7), with a continuous generator h. Since $P_n([a,b])\hookrightarrow C\Phi BV(I)$, by Corollary 1 of [12], and $(C\Psi BV(I),\|\cdot\|_\Psi)$ is a closed subspace of $(\Psi BV(I),\|\cdot\|_\Psi)$, i.e., $C\Psi BV(I)\hookrightarrow\Psi BV(I)$, using Theorem 2 and Definition 3, we get a required form (20) of an operator K, by continuity of h with respect to each variable.

In contrary, since every operator defined by (20) has memory property and the space $(C\Psi BV(I),\|\cdot\|_\Psi)$ is a Banach algebra, the proof is completed. □

As a by-product, we obtain the following

Corollary 3. The pair $(C\Phi BV(I),C\Psi BV(I))$ of the normed spaces $(C\Phi BV(I),\|\cdot\|_\Phi)$ and $(C\Psi BV(I),\|\cdot\|_\Psi)$ has the uniform Matkowski property.

5. Conclusion

Applying the main result of [12], we prove that every operator with memory K mapping a Banach space $C\Phi BV(I)$ of continuous functions of bounded variation in the sense of Schramm into another space $C\Psi BV(I)$ is a Nemytskij composition operator with the continuous generating function h. By choosing the appropriate Schramm sequences $\Phi = (\varphi_i)_{i=1}^{\infty}$this, among others, generalizes previous results [3] and [4]. Moreover, under the additional assumption that the operators with memory are uniformly bounded, we observe that operators of such a type must be of the form $K(f) = \alpha \cdot f + \beta$, where α and β are the elements from the space $C\Psi BV(I)$. As a by-product, we obtain that the pair $(C\Phi BV(I), C\Psi BV(I))$ of the normed spaces $(C\Phi BV(I), \|\cdot\|_\Phi)$ and $(C\Psi BV(I), \|\cdot\|_\Psi)$ has the uniform Matkowski property.

References

[1] Lichawski, K., Matkowski, J., & Miś, J. (1989). Locally defined operators in the space of differentiable functions. *Bull. Polish Acad. Sci. Math.*, *37*, 315-325.

[2] Wróbel, M. (2010). Locally defined operators and a partial solution of a conjecture. *Nonlinear Anal. TMA*, *72*, 495-506.

[3] Wróbel, M. (2013). Locally defined operators in the space of functions of bounded φ-variation. *Real Anal. Exch.*, *38*(1), 79-92.

[4] Wróbel, M. (2021). The form of locally defined operators in Waterman spaces. *Math. Slovaca*, *71*(6), 1529-1544.

[5] Schramm, M., & Waterman, D. (1982). Absolute convergence of Fourier series of functions of ΛBV_p and $\Phi\Lambda BV$. *Acta Math. Acad. Sci. Hung.*, *40*(3-4), 273-276.

[6] Brunner, H. (2017). *Volterra Integral Equations*. Cambridge University Press.

[7] Kmit, I. (2021). C^1-smoothness of Nemytskii operators on Sobolev-type spaces of periodic functions. *Commentat. Math. Univ. Carol.*, *52*(4), 507-517.

[8] Grützner, S., & Muntean, A. (2021). Identifying processes governing damage evolution in quasi--static elasticity, Part 1 - Analysis. *Advances in Mathematical Sciences and Applications*, *30*(2), 305-334.

[9] Krämer, R, & Mathé, P. (2008). Modulus of continuity of Nemytskii operators with application to the problem of option pricing. *Journal of Inverse and Ill-posed Problems*, *16*(5), 435-461.

[10] Appell, J., Banaś, J., & Merentes, N. (2014). *Bounded Variation and Around*. Berlin, Boston: De Gruyter.

[11] Guerrero, J.A, Leiva, H., Matkowski, J., & Merentes, N. (2010). Uniformly continuous composition operators in the space of bounded φ-variation functions. *Nonlinear Anal.*, *72*, 3119-3123.

[12] Wróbel, M. (2023). Schramm spaces and composition operators. *J. Appl. Math. Comput. Mech.*, *22*(2), 87-98.

[13] Matkowski, J. (2011). Uniformly bounded composition operators between general Lipschitz function normed spaces. *Topol. Methods Nonlinear Anal.*, *38*(2), 395-405.