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AN ANALYTICAL STUDY OF COUPLE STRESS FLUID THROUGH A SPHERE WITH AN INFLUENCE OF THE MAGNETIC FIELD

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Abstract. The present work concerns to study of the steady, axisymmetric slow flow of couple stress fluid through a rigid sphere in the transverse magnetic field. Boundary conditions on the sphere surface are the zero couple stress condition and tangential slip condition. The stream function, vorticity vector, and pressure term are obtained. The drag acting on the sphere in the presence of MHD is calculated. Here, we graphically represented the Hartmann number, couple stress, and slip parameters effect on the drag coefficient. Some well-known results of the drag are deduced.

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Keywords: *creeping flow, sphere, magnetic effect, couple stress fluid, drag*

1. Introduction

The study of couple stress fluid attracts the attention of many investigators due to its application in engineering, extrusion of polymer fluid, industry, animal blood, and solidification of liquid crystals. The fluids that disobey the linear relationship between the rate of strain and stress are called non-Newtonian fluids. The couple stress fluid belongs to the non-Newtonian fluid category, which has distinct features, such as size dependent effect. Stokes [1] introduced the theory of couple stress fluid in 1966. He considers the couple stress fluid at kinematic level to have no microstructure so that kinematics of motion is obtained by the velocity field only [2]. Devakar et al. [3] obtained the analytical solution for flow of the couple stress fluid with slip effect. Srinivasacharya et al. [4] investigated the couple stress fluid flow through two parallel porous plates. Ashmawy [5] investigated the drag exerted by couple stress fluid on the slip sphere. He noticed that whenever the slip parameter increases, drag increases. Aparna et al. [6] obtained oscillating flow of a couple stress fluid through a permeable sphere. Krishna Prasad and Priya [7] examined the slip effect on the sphere in a porous medium filled with couple stress fluid. Pan et al. [8] studied the transitions and bifurcations in couple stress fluid saturated porous media using a thermal non-equilibrium model. Alsudais et al. [9] investigated the creeping flow of couple stress fluid confined between two eccentric spheres.

The consistent couple stress theory was introduced by Hadjesfandiari and Dargush [10–13]. In this theory, the couple stress tensor is skew-symmetric. The consistent couple stress theory is one of the new and important theories in the world of technology. Many researchers are attracted to doing research using this theory. Subramaniam and Mondal [14] studied dynamics and rheology of the linear Maxwell viscoelastic fluid model with the effect of couple stress. Karami et al. [15] investigated the solution of the flow of viscous nano fluid in a tapered artery using the consistent the couple stress theory.

The magnetic properties and behavior of electrically conducting fluids are studied in field of Magnetohydrodynamics (MHD). Hannes Alfven introduced the field of MHD [16]. Hannes Alfven [16] also discussed that when a liquid is placed in the magnetic field, every motion of liquid generates an E.M.F., which produces an electric current. This current gives mechanical force which changes the state of motion of the liquid. Many researchers have been attracted to the study of magnetohydrodynamics because of its vast applications, such as solar physics, geophysics, industry, and fusion energy research. MHD influences many man-made and natural flows. Globe [17], and Gold [18] have obtained the effect of the magnetic field in an annular channel and pipe flow. Saad [19] examined the flow of viscous fluid through a porous sphere in a magnetic field based on the cell model technique. Sherief et al. [20] observed the pipe flow of magneto-micropolar fluids with slip influence. Krishna Prasad and Priya [21] investigated the Stokes flow through a slip sphere in the cell model in the MHD. El-Sapa and Faltas [22] studied the mobilities of two spherical particles immersed in a magneto-micropolar fluid.

The purpose of this work is to investigate the effect of a transverse magnetic field on the couple stress fluid flow through a sphere. We consider that flow is an axisymmetric steady type. Boundary conditions consist of the zero couple stress condition and the slip condition. Stream function and pressure expression are solved. The drag force acting on the rigid sphere in MHD is obtained, and some previous well-known cases which are essential for this work are discussed. The drag coefficient depends on the Hartmann number, couple stress, slip, and couple stress viscosity ratio parameters, and their effects are presented graphically.

2. Mathematical formulation

Consider the steady axisymmetric incompressible flow of magneto couple stress fluid through a sphere of radius $r = a$ which is fixed in a uniform velocity U. The magnetic field is applied in transverse direction to the fluid velocity vector $\vec{q}$, as shown in Figure 1. Here, we consider small Reynold's number $Re = \dfrac{Ua}{\nu}$, i.e., only viscous term is exist. The following assumptions are [19, 21]: the fluid will not have any micro structure, there is no body moment and body force, the magnetic field direction is normal to the flow, the $Re_m = Ua\mu_h\sigma$ magnetic Reynold's number is sufficiently small, neglected the induced magnetic field, there is no external electric fluid, Lorentz force $\vec{F}$ is defined by $\vec{F} = \vec{H} \times \vec{J}$, where: μ_h – the magnetic permeability, σ – the electric conductivity, $\vec{H}$, $\vec{J}$ represent the magnetic induction vector, and electric current density, respectively $\vec{F} = \mu_h^2\sigma(\vec{q} \times \vec{H}) \times \vec{H}$.

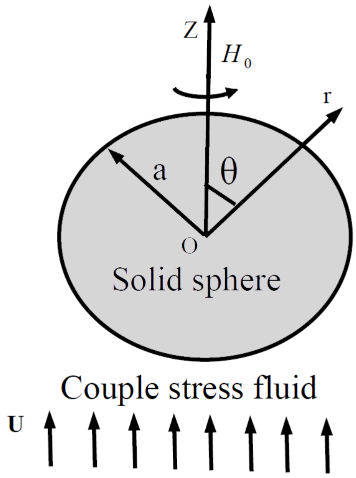

Fig. 1. Sketch of the problem

The governing equations of an incompressible couple stress fluid with the effect of magnetic field are given by [2, 19, 21]

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

$$\eta\nabla \times \nabla \times \nabla \times \nabla \times \vec{q} - \mu_h^2\sigma(\vec{q} \times \vec{H}) \times \vec{H} + \mu\nabla \times \nabla \times \vec{q} + \nabla p = 0, \tag{2}$$

where p, $\vec{q}$, μ, μ_h^2, $\vec{H}$, σ represent the pressure, velocity, viscosity coefficient, magnetic permeability of the fluid, magnetic induction vector, electric conductivity. The couple stress viscosity are η and η', and they follow this restriction $\eta \geq \eta'$ [2].

The expression of stress and couple stress are [2]

$$t_{ij} = -p\delta_{ij} + 2\mu d_{ij} - \frac{1}{2}e_{ijk}m_{sk,s,}, \tag{3}$$

Where δ_{ij} is the Kronecker delta, and e_{ijk} is the alternating tensor

$$m_{ij} = m\delta_{ij} + 4(\eta'\omega_{i,j} + \eta\omega_{j,i}), \tag{4}$$

Where $\omega_{i,j}$ presents the spin tensor.

The vorticity vector ω is defined as [5]

$$\omega_i = \frac{1}{2}e_{ijk}q_{k,j}. \tag{5}$$

The deformation rate tensor d_{ij} is given as [2]

$$d_{ij} = \frac{1}{2}(q_{i,j} + q_{j,i}). \tag{6}$$

With the help of non-dimensional variables, we reformed the governing equations into non-dimensionalise equations

$$r = a\tilde{r}, \quad \nabla = \frac{\tilde{\nabla}}{a}, \quad \vec{q} = U\tilde{\vec{q}}, \quad \vec{H} = H_0\tilde{\vec{H}}, \quad p = \frac{\mu U}{a}\tilde{p}.$$

Inserting these terms in Eq. (1) and Eq. (2) and then neglecting the tildes

$$\nabla \cdot \vec{q} = 0, \tag{7}$$

$$\nabla p + \nabla \times \nabla \times \vec{q} - M^2(\vec{q} \times \vec{H}) \times \vec{H} + S^{-2}\nabla \times \nabla \times \nabla \times \nabla \times \vec{q} = 0, \tag{8}$$

Where length dependent parameter is $S = \sqrt{\frac{\mu a^2}{\eta}}$, if S tends to infinity, Eq. (8) represent the modified Stokes equation in presence of Lorentz's force for non-polar fluid. $M = \sqrt{\frac{\mu_h^2 H_0^2 \sigma a^2}{\mu}}$ is the Hartmann number. If $M = 0$, Eq. (8) is Stokes approximation of the couple stress fluid.

Let (r, θ, ϕ) represent spherical polar coordinate system, as we know that the fluid flow is axisymmetric type, i.e., all the quantities do not depend on ϕ. Therefore, the velocity vector $\vec{q}$ is

$$\vec{q} = q_r(r, \theta)e_r + q_\theta(r, \theta)e_\theta. \tag{9}$$

The velocity components are expressed in the stream function which satisfies continuity equation,

$$q_r = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad q_\theta = \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r}. \tag{10}$$

Omitting the pressure from Eq. (7) using Eq. (10), we get

$$E^2\left(E^2 - \ell_1^2\right)\left(E^2 - \ell_2^2\right)\psi = 0, \tag{11}$$

Where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta},$$

$$\ell_1^2 = \frac{S^2 + S\sqrt{S^2 - 4M^2}}{2},$$

$$\ell_2^2 = \frac{S^2 - S\sqrt{S^2 - 4M^2}}{2}.$$

3. Analytical solution

The stream function of linear partial differential Eq. (11) is given by [7]

$$\psi = \frac{1}{2}\left(r^2 + Ar^{-1} + B\sqrt{r}K_{3/2}(\ell_1 r) + C\sqrt{r}K_{3/2}(\ell_2 r)\right)\sin^2\theta, \tag{12}$$

Where $K_{3/2}(\cdot)$ is the modified Bessel function of the 2^{nd} kind of order $3/2$. Arbitrary constants are A, B, and C.

By substituting Eq. (12) into Eq. (10), we get

$$q_\theta = \frac{1}{-}\left[2\frac{\;}{2}Ar^{-3} - Br^{-\frac{3}{2}}\left(\ell_1 r K_{1/2}(\ell_1 r) + K_{3/2}(\ell_1 r)\right) - Cr^{-\frac{3}{2}}\left(\ell_2 r K_{1/2}(\ell_2 r) + K_{3/2}(\ell_2 r)\right)\right]\sin\theta, \tag{13}$$

$$q_r = -\left[1 + Ar^{-3} + Br^{-\frac{3}{2}}K_{3/2}(\ell_1 r) + Cr^{-\frac{3}{2}}K_{3/2}(\ell_2 r)\right]\cos\theta, \tag{14}$$

The expression for the vorticity vector is

$$\omega_\phi = \frac{1}{4}\left[Br^{-\frac{1}{2}}\ell_1^2 K_{3/2}(\ell_1 r) + Cr^{-\frac{1}{2}}\ell_2^2 K_{3/2}(\ell_2 r)\right]\sin\theta, \tag{15}$$

The pressure is

$$p = M^2\left(r - \frac{1}{2}Ar^{-2}\right)\cos\theta. \tag{16}$$

4. Boundary conditions

In order to obtain unknowns A, B, and C, appropriate conditions are used. Boundary conditions for the proposed model are vanishing of normal component of the velocity, tangential slip condition, zero couple stresses [2, 5, 7, 21, 23, 24].

On the surface of sphere $r = a$

- Vanishing of normal velocity

$$q_r = 0, \tag{17}$$

- Slip condition

$$\beta_1 q_\theta = t_{r\theta}, \tag{18}$$

- Zero couple stress on the boundary

$$m_{r\phi}=0, \tag{19}$$

Where β_1 represent the slip coefficient, which is dependent on the fluid nature and solid surface. If $\beta_1\to\infty$, the sphere surface is a no-slip. If $\beta_1=0$, the sphere surface is a perfect slip.

The couple stress components are

$$m_{r\phi}=-r^{-\frac{3}{2}}\left[\left((2\eta+\eta')\ell_1^2K_{3/2}(\ell_1 r)+\eta\ell_1^3 rK_{1/2}(\ell_1 r)\right)B\right.\\ \left.+\left((2\eta+\eta')\ell_2^2K_{3/2}(\ell_2 r)+\eta\ell_2^3 rK_{1/2}(\ell_2 r)\right)C\right]\sin\theta, \tag{20}$$

$$m_{\phi r}=-r^{-\frac{3}{2}}\left[\left((\eta+2\eta')\ell_1^2K_{3/2}(\ell_1 r)+\eta'\ell_1^3 rK_{1/2}(\ell_1 r)\right)B\right.\\ \left.+\left((\eta+2\eta')\ell_2^2K_{3/2}(\ell_2 r)+\eta'\ell_2^3 rK_{1/2}(\ell_2 r)\right)C\right]\sin\theta, \tag{21}$$

$$m_{\theta\phi}=r^{-\frac{3}{2}}\left[\left((\eta-\eta')\ell_1^2K_{3/2}(\ell_1 r)\right)B+\left((\eta-\eta')\ell_2^2K_{3/2}(\ell_2 r)\right)C\right]\cos\theta, \tag{22}$$

The stress components are

$$t_{rr}=\left[-M^2\left(r-\frac{1}{2}Ar^{-2}\right)+6Ar^{-4}+r^{-\frac{5}{2}}\left(6K_{3/2}(\ell_1 r)+2\ell_1 rK_{1/2}(\ell_1 r)\right)B\right.\\ \left.+r^{-\frac{5}{2}}\left(6K_{3/2}(\ell_2 r)+2\ell_2 rK_{1/2}(\ell_2 r)\right)C\right]\cos\theta, \tag{23}$$

$$t_{r\theta}=\frac{1}{2}\left[6Ar^{-4}+r^{-\frac{5}{2}}\left(\left(6+r^2M^2\right)K_{3/2}(\ell_1 r)+2\ell_1 rK_{1/2}(\ell_1 r)\right)B\right.\\ \left.+r^{-\frac{5}{2}}\left(\left(6+r^2M^2\right)K_{3/2}(\ell_2 r)+2\ell_2 rK_{1/2}(\ell_2 r)\right)C\right]\sin\theta, \tag{24}$$

Applying the boundary conditions (17)-(19), we have

$$A=-\sqrt{a}((J_1L_1+J_2L_2)L_3+(J_3L_1+J_4L_2)L_4)\Delta^{-1}, \tag{25}$$

$$B=a^{-1}3\ell_4^2(a\beta_1+2)((\tau+2)L_4+\ell_4L_3)\Delta^{-1}, \tag{26}$$

$$C=-a^{-1}3\ell_3^2(a\beta_1+2)((\tau+2)L_2+\ell_3L_1)\Delta^{-1}, \tag{27}$$

Where

$$
\begin{aligned}
&\Delta = a^{-\frac{5}{2}}((J_5L_2+J_6L_1)L_4+(J_7L_2+J_1L_1)L_3),\\
&J_1 = \ell_3\ell_4(a\beta_1+2)(\ell_4^2-\ell_3^2),\\
&J_2 = \ell_4(\ell_4^2(3(a\beta_1+2)+a^2M^2)-\ell_3^2(a\beta_1+2)(\tau+2)),\\
&J_3 = \ell_3(\ell_4^2(a\beta_1+2)(\tau+2)-\ell_3^2(3(a\beta_1+2)+a^2M^2)),\\
&J_4 = (\tau+2)(\ell_4^2-\ell_3^2)(3(a\beta_1+2)+a^2M^2),\\
&J_5 = a^2M^2(\tau+2)(\ell_4^2-\ell_3^2),\\
&J_6 = \ell_3(\ell_4^2(\tau+2)(a\beta_1+2)-a^2M^2\ell_3^2),\\
&J_7 = -\ell_4(\ell_3^2(\tau+2)(a\beta_1+2)-a^2M^2\ell_4^2),\\
&L_1 = K_{1/2}(\ell_3), \quad L_2 = K_{3/2}(\ell_3), \quad L_3 = K_{1/2}(\ell_4), \quad L_4 = K_{3/2}(\ell_4),\\
&\ell_3 = \ell_1 a, \quad \ell_4 = \ell_2 a, \quad \tau = \frac{\eta'}{\eta}.
\end{aligned}
$$

5. Drag force

The drag exerted by couple stress fluid on the sphere in the presence of magnetic and slip effect is evaluated using the simple formula

$$F_z = 2\pi a^2 \int_0^{\pi} r^2 \left(t_{rr}\cos\theta - t_{r\theta}\sin\theta\right)\Big|_{r=a} \sin\theta d\theta, \tag{28}$$

Inserting the values of Eq. (23) and Eq. (24) values in this formula, we have

$$F_z = \frac{2}{3}\pi\mu aUM^2\left(-2a^3 + A - 2a\sqrt{a}L_2B - 2a\sqrt{a}L_4C\right). \tag{29}$$

6. Special cases

Case 1: If $M = 0$ i.e., $\ell_2 = 0$ and $\ell_1 = S$, the drag exerted by the couple stress fluid through a sphere in the absence of MHD is given by

$$F_z = -6\pi\mu aU\left(1 + \frac{M_1(\ell_5+1)-\beta_2}{M_1+\beta_2(\ell_5+3)}\right), \tag{30}$$

Where

$$\beta_2 = aS, \quad M_1 = \frac{\tau+2}{\tau+\beta_2+2}, \quad \ell_5 = \frac{a\beta_1}{\mu}.$$

This result is same as Ashmawy [5].
Case 2: If $\ell_5 \to \infty$ in Eq.(30), i.e., the drag acting on the no-slip sphere is

$$F_z = -6\pi\mu aU\left(1+\frac{M_1}{\beta_2}\right). \tag{31}$$

This result agrees with V.K. Stokes' result [25].
Case 3: Drag force exerted by the consistent couple stress fluid on the sphere in the presence of tangential slip and MHD, i.e., $\tau = -1$, we get

$$F_z = -2\pi\mu aUM^2a^2\left((J_1L_3+J_3L_4)L_1+(J_2L_3+J_4L_4)L_2\right)\Delta_1^{-1}, \tag{32}$$

Where

$$\Delta_1 = (J_1L_3+J_6L_4)L_1+(J_5L_4+J_7L_3)L_2.$$

Case 4: If $M = 0$ i.e., $\ell_2 = 0$ and $\ell_1 = S$ in Eq. (32), the drag acting on the sphere in the absence of MHD is

$$F_z = -6\pi\mu aU\left(1+\frac{(\ell_5+1)-\beta_2(\beta_2+1)}{1+\beta_2(\ell_5+3)(\beta_2+1)}\right). \tag{33}$$

Case 5: If $\ell_5 \to \infty$ in Eq. (33), the drag force reduces to

$$F_z = -6\pi\mu aU\left(\frac{\beta_2^2+\beta_2+1}{\beta_2(\beta_2+1)}\right). \tag{34}$$

This result match with the result of Hadjesfandiari and Dargush [10].
Case 6: If $\beta_2 \to \infty$, i.e., $\eta \to 0$ in (30), the drag is

$$F_z = -6\pi\mu aU\left(\frac{\ell_5+2}{\ell_5+3}\right). \tag{35}$$

Eq. (35) represents the drag acting on the slip sphere in the Newtonian fluid.
Case 7: If $\beta_2 \to \infty$ and $\ell_5 \to \infty$ in Eq. (30), the drag experienced by the viscous fluid on the no-slip sphere (Stokes law) is obtained as

$$F_0 = -6\pi\mu aU. \tag{36}$$

7. Results and discussion

The normalized force D_N defined as the ratio of drag acting on the sphere in couple stress fluid with influence of a transverse magnetic field and tangential slip to the drag F_0 acting on the no-slip sphere in a viscous fluid. The variation of D_N with respect to different parameters is shown graphically in Figures 2-5.

The normalized drag force D_N is a function of various parameters as follows:
(i) Slip parameter: β_1,

(ii) Couple stress viscosity ratio parameter: τ,
(iii) Couple stress parameter: S,
(iv) Hartmann number (magnetic parameter): M.

Figure 2 represents the variation of D_N versus the slip β_1 for varying values of S with fixed value of $M = 0.1$ and $\tau = 1$. We have seen that when S increases, the force D_N decreases, and after $S = 1$, the difference between the variation of drag force corresponding to other S is very small.

The effect of τ on the normalized drag force D_N is discussed in Figure 3. It shows that a decrease in values of couple stress viscosity ratio parameter τ, results a decreasing value of drag force. Figure 4 presents the variation of D_N against the couple stress parameter S for varying values of β_1 with $M = 0.001$ and $\tau = 1$. We observed that when we increase the value of slip parameter β_1, the value of the drag parameter slightly increases. Figure 5 shows the variation of drag force D_N against slip parameter β_1 for different values of the Hartmann number M. When the value of Hartmann number M slightly increases, it results in an increased value in the drag D_N.

When $\tau = -1$ i.e., $\eta' = -\eta$ represents the consistent couple stress theory [10–13]. In this case, the normalized drag force D_N depends on the slip parameter β_1, couple stress parameter S, and magnetic parameter M. Figure 6 presents the variation of D_N against the slip parameter β_1 for varying parameter S with $\tau = -1$ and $M = 0.1$. We have noticed that increasing the values of S, the drag force decreases. Figure 7 shows the variation of D_N against couple stress parameter S for varying values of the β_1 with $\tau = -1$ and $M = 0.001$. It is observed that when we increase the values of β_1, the drag force slightly increases due to the influence of couple stress viscosity ratio parameter $\tau = -1$. The variation of D_N against slip parameter β_1 for varying values of the Hartmann number M with $\tau = -1$ and $S = 3$ is discussed in Figure 8. Additionally, we found that increasing the values of M results in an increase in the drag force.

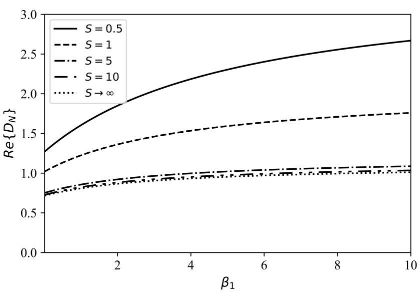

Fig. 2. Drag D_N variation versus β_1 for $M = 0.1$ and $\tau = 1$

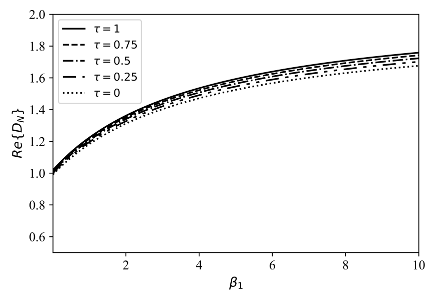

Fig. 3. Drag D_N variation against β_1 for $M = 0.1$ and $S = 1$

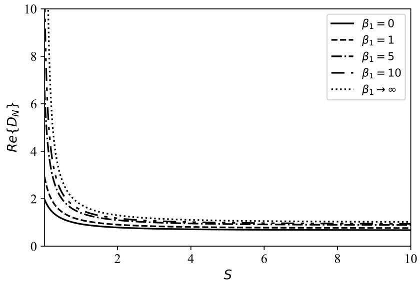

Fig. 4. Drag D_N variation versus S for $M = 0.001$ and $\tau = 1$

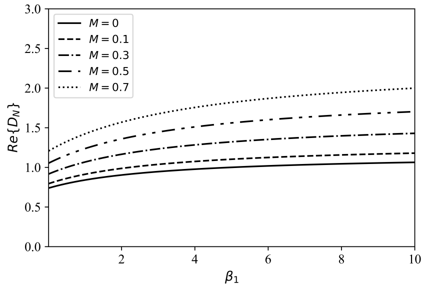

Fig. 5. Drag D_N variation against β_1 for $S = 3$ and $\tau = 1$

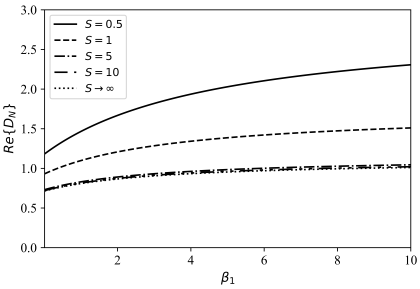

Fig. 6. Drag D_N variation versus β_1 for $M = 0.1$ and $\tau = -1$

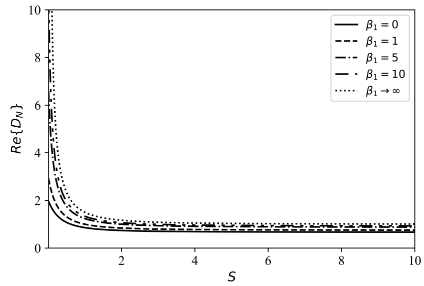

Fig. 7. Drag D_N variation versus S for $M = 0.001$ and $\tau = -1$

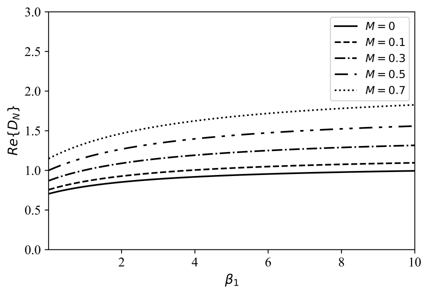

Fig. 8. Drag D_N variation against β_1 for $S = 3$ and $\tau = -1$

8. Conclusions

The couple stress fluid flow through a sphere fixed with the influence of a tangential slip and magnetic effect is solved analytically. The drag exerted by the couple stress fluid on the rigid sphere in the presence of a transverse magnetic field is obtained. We discussed some well-known cases from the past. We noticed from the figure that when couple stress parameter increases, normalized drag force decreases. The normalized drag force increases as the magnetic parameter increases. The fluid velocity decreases in the presence of the Lorentz force. Due to the Lorentz force effect, when we increase the value of M, the result is a normalized increase in drag force. For consistent couple stress fluid, we observed that when a couple stress parameter increases, the normalized drag decreases, and when the slip parameter increases, drag increases.

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