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USING SHEHU INTEGRAL TRANSFORM TO SOLVE FRACTIONAL ORDER CAPUTO TYPE INITIAL VALUE PROBLEMS

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Abstract. In the present research analysis, linear fractional order ordinary differential equations with some defined condition (s) have been solved under the Caputo differential operator having order $\alpha > 0$ via the Shehu integral transform technique. In this regard, we have presented the proof of finding the Shehu transform for a classical nth order integral of a piecewise continuous with an exponential order function which leads towards devising a theorem to yield exact analytical solutions of the problems under investigation. Varying fractional types of problems are solved whose exact solutions can be compared with solutions obtained through existing transformation techniques including Laplace and Natural transforms.

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1. Introduction

The area called fractional calculus nowadays seems to be a vibrant area for many researchers and scientists from all around the world. This field was indeed born around 1695 due to an exchange of ideas between two renowned mathematicians of that time. The meaning of a half derivative of a function was asked but no adequate response was received. However, it was predicted that this area of mathematics will one day be more popular than its counterpart known as classical calculus wherein we use derivatives and integrations of an integer-order whereas fractional calculus takes all arbitrary order differentials and integrals.

Unlike classical differential equations, there is no universally accepted physical and geometrical interpretation of fractional-order derivatives and integrals, however, such fractional-order mathematical models are proved to be extremely useful in various fields of science and engineering. Therefore, much attention has been given to tackle the problems arising from these fields. For instance, Reserchers in [1, 2] used fractional-order derivatives to understand the copmlex behaviour of some chaotic models. In the field of mathematical epidemiology, many authors have recently fractionalyzed classical systems related with various types of diseases and proved their efficacy with the help of real statistical data [3–7]. Khan in [8] studied blood flow problems in the form of nano-fluids under a newly-proposed fractional-order operator with a non-local nature and a non-singular kernel. Apart from science and technology, the field of classical and contemporary fractional calculus is also found to be useful in social interactions among humans which has been described in [9] with the help of a dynamical mathematical model under a fractional operator with the Mittag--Leffler type kernel. As long as computer science is related, the models depending upon fractional-order derivatives are found to be useful to understand the ways for the spread of computer bugs and viruses [10]. Imran et al. in [11] proved the better performance of Caputo fractional order derivatives to analyze the fluid flow problems with regard to non-dimensional temperature, concentration, and velocity fields.

Above all, there is not even a single field of study wherein fractional calculus has not been found beneficial to design and comprehend the complex behavior of deterministic and stochastic systems, for example see, [12–18] and most of the references cited therein. In short, many real world problems such as natural and biological ones are touched upon with the tools of fractional calculus making it a burning topic of today's era. One of the major reasons for its popularity is its capability to deal with very complex and anomalous systems with memory and hereditary characteristics.

In their attempt to find the solution of dynamical models, many scientists come across nonlinearities in their models which create a hindrance towards finding the closed form or analytical solutions. On the other hand, the models with linear components are comparatively easy to deal with. In this regard, one does not need any sort of numerical technique to get the exact solution of the system under investigation. Therefore, the purpose is served with the help of integral transform techniques. The present literature is rich enough with different kinds of integral transforms, namely, Laplace transform, Fourier transform, Mellin transform, Laplac-Carson transform, Atangana-Kilicman transform, Elzaki transform, Yang transform, $\mathbb{M}$-transform, Mohand transform, Aboodh transform, and the most recently proposed one known as the Shehu transform [19–24].

The major goal of the present research analysis is to test the recently proposed integral transform with the name of Shehu integral transform for solving fractional order initial value problems under the Caputo differential operator with α being the order of the system under analysis. Different types of initial value problems are solved to observe the efficacy of the Shehu transform for fractional type of models. In this regard, the fractional order parameter α is taken to be any positive real constant.

2. Mathematical preliminaries

To understand the research analysis presented in this paper, it is necessary to study some basic concepts of fractional order calculus and the Shehu integral transform technique. In this connection, a few important existing concepts are provided below.

Definition 1 [19] The Shehu integral transform for a piecewise continuous function $f(t)$ with exponential order P is defined over the set of functions:

$$\mathbb{A} = \left\{ f(t) : \exists\, P, a_1, a_2 > 0,\ |f(t)| < P \exp\left(\frac{|t|}{a_i},\right), \text{if } t \in (-1)^i \times [0,\infty) \right\}, \quad (1)$$

by the following integral

$$S[f(t)] = F(s,u) = \int_0^\infty \exp\left(-\frac{s}{u}t\right) f(t)dt. \quad (2)$$

Definition 2 [25] The fractional order derivative called Caputo derivative of function $f(t)$ having order $\alpha > 0$ is defined by the following integral:

$${}^C D^{\alpha}_{0,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \qquad n-1 < \alpha \le n \in \mathbb{N}. \quad (3)$$

Definition 3 [25] The fractional order integral called the Riemann-Liouville integral for function $f(t)$ having order $\alpha > 0$ is defined by the following equation:

$$J^{\alpha}_{0,t}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \quad t > 0, \quad (4)$$

where $\Gamma(\cdot)$ is called the Euler gamma function. □

Theorem 1 *[19] Consider $f^{(n)}(t)$ be the nth order classical derivative of the function $f(t) \in \mathbb{A}$, then its Shehu integral transform is given by the following formula:*

$$S[f^{(n)}(t)] = \left(\frac{s}{u}\right)^n F(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-k-1} f^{(k)}(0), \quad n \in \mathbb{N}. \quad (5)$$

Lemma 1 *[25] The fractional order derivative Caputo and the fractional order integral Riemann-Liouville are related to each other by the following equation:*

$${}^C D^{\alpha}_{0,t} f(t) = J^{n-\alpha}_{0,t}\left[D^n f(t)\right], \quad (n-1) < \alpha < n, n \in \mathbb{N}. \quad (6)$$

3. Research methodology

In this part of the paper, a major concern is to devise a tool which may help us to solve fractional order initial value problems defined under the Caputo type differential operator having order $\alpha > 0$. In this connection, recently proposed transform technique called the Shehu integral transform is tested to serve the purpose. As an initial step, an important property for finding the Shehu integral transform for classical integral of a function $f(t)$ is derived. Later, this property along with some basic concepts as presented in section 2 are used to prove a new theorem that will consequently be used to get the exact analytical solutions for the problems under investigation.

Theorem 2 *If $F(s,u)$ is the Shehu integral transform of the function $f(t)$ then the Shehu integral transform for the classical integral $f(t)$ can be defined by the following relation:*

$$S\left\{\int_0^t f(u)du\right\} = \frac{u}{s}F(s,u). \tag{7}$$

PROOF Using the definition of the Shehu integral transform for $f(t)$, one obtains

$$S\left\{\int_0^t f(u)du\right\} = \int_0^\infty \exp(-\frac{s}{u}t)dt.\int_0^t f(u)du. \tag{8}$$

Employing the integration by parts technique, we reach at the following equations:

$$S\left\{\int_0^t f(u)du\right\} = \int_0^t f(u)du\int_0^\infty \exp\left(\frac{-s}{u}t\right)dt - \int_0^\infty\left\{\frac{d}{dt}\int_0^t f(u)du\int_0^\infty \exp\left(\frac{-s}{u}t\right)dt\right\}dt,$$

$$S\left\{\int_0^t f(u)du\right\} = \int_0^t - f(u)du\frac{u}{s}\exp\left(\frac{-s}{u}t\right)\Bigg|_0^\infty + \frac{u}{s}\int_0^\infty \exp\left(\frac{-s}{u}t\right)f(t)dt,$$

$$S\left\{\int_0^t f(u)du\right\} = \frac{u}{s}F(s,u).$$

Similarly, $S\left\{\int_0^t\int_0^t f(u)du^2\right\} = \left(\frac{u}{s}\right)^2 F(s,u)$. Continuing in this way, one obtains the following generalized formula:

$$S\left\{\int_0^t \dots \int_0^t f(u)du^n\right\} = \left(\frac{u}{s}\right)^n F(s,u). \tag{9}$$

Next, we present a theorem which has been devised bearing in mind that it would be capable enough to get the exact analytical solutions for fractional order initial value problems under the Caputo type operator having order $\alpha > 0$.

Theorem 3 *If $F(s,u)$ is the Shehu integral transform of function $f(t)$ then the Shehu transform of fractional order derivative for $f(t)$ under the Caputo type operator* *having order $\alpha > 0$ is proposed as follows:*

$$S\{{}^{C}D^{\alpha}_{0,t}f(t)\} = \left(\frac{s}{u}\right)^{\alpha} F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-k-1} f^{(k)}(0). \tag{10}$$

PROOF From the Lemma 1, one can write the following:

$${}^{C}D^{\alpha}_{0,t}f(t) = J^{n-\alpha}_{0,t}\Big[D^{n}f(t)\Big]. \tag{11}$$

Taking Shehu transform on both sides, one obtains the following:

$$S\{{}^{C}D^{\alpha}_{0,t}f(t)\} = S\{J^{n-\alpha}_{0,t}(D^{n}f(t))\}. \tag{12}$$

Now using the Shehu transform for the integral of a function as proposed in 2, one obtains the following:

$$\begin{aligned} S\{{}^{C}D^{\alpha}_{0,t}f(t)\} &= \left(\frac{u}{s}\right)^{n-\alpha} S\{(D^{n}f(t))\}, \\ S\{{}^{C}D^{\alpha}_{0,t}f(t)\} &= \left(\frac{u}{s}\right)^{n-\alpha}\Big[\left(\frac{s}{u}\right)^{n} F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1} f^{(k)}(0)\Big], \\ S\{{}^{C}D^{\alpha}_{0,t}f(t)\} &= \left(\frac{s}{u}\right)^{\alpha} F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-k-1} f^{(k)}(0), \quad n \in \mathbb{N},\ \alpha > 0. \end{aligned} \tag{13}$$

This completes the required proof for the Shehu integral transform for fractional order derivatives under the Caputo operator having order $\alpha > 0$. This proposed theorem will ultimately help us to solve the fractional order linear initial value problems defined with the Caputo type operator.

4. Results and discussion

In this section, varying types of fractional order initial value problems under the Caputo derivative of order $\alpha > 0$ are taken to test the performance of the Shehu integral transform technique with the help of the above-proposed theorem 3 which is basically obtained for finding the Shehu transform of fractional order derivatives. Some non-homogeneous initial value problems with fractional order $\alpha = 1/2$ and $\alpha = 3/2$ with Caputo operator are analytically solved via the Shehu transform technique. The obtained results in terms of elementary mathematical functions agree well with those obtained via integral transforms including the Laplace and Natural. Hence, the Shehu integral transform is considered to be an additional tool for solving fractional order Caputo type continuous dynamical systems. It is also worth noting that this transformation technique is used for finding exact analytical solutions for classical differential equations but in the present research analysis the technique, for the first time, is employed on the fractional type of problems.

Example 1 Consider the following linear fractional order in-homogeneous initial value problem with $\alpha = 3/2$:

$$D^2 f(t) + {}^C D_{0,t}^{\frac{3}{2}} f(t) + f(t) = t, \qquad f(0) = 0, f'(0) = 1. \tag{14}$$

Using the Shehu integral transform, one obtains the following:

$$\begin{aligned}
&S\{D^2 f(t)\} + S\{{}^C D_{0,t}^{\frac{3}{2}} f(t)\} + S\{f(t)\} = S\{t\},\\
&\left(\frac{s}{u}\right)^2 F(s,u) - 1 + \left(\frac{s}{u}\right)^{\frac{3}{2}} F(s,u) - \left[\left(\frac{s}{u}\right)^{\frac{1}{2}} \times 0 + \left(\frac{s}{u}\right)^{\frac{-1}{2}}\right] + F(s,u) = \left(\frac{u}{s}\right)^2,\\
&F\left\{\left(\frac{s}{u}\right)^2 + \left(\frac{s}{u}\right)^{\frac{3}{2}}\right) + 1\right\} = \left(\frac{u}{s}\right)^2 \left\{1 + \left(\frac{s}{u}\right)^2 + \left(\frac{s}{u}\right)^{\frac{3}{2}}\right\},\\
&F(s,u) = \left(\frac{u}{s}\right)^2.
\end{aligned} \tag{15}$$

Taking the inverse Shehu transform, the exact analytical solution is obtained as:

$$f(t) = t. \tag{16}$$

Example 2 Consider the following linear fractional order in-homogeneous initial value problem with $\alpha \in (0,1]$:

$${}^C D_{0,t}^{\alpha} f(t) + f(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + t^2, \qquad f(0) = 0. \tag{17}$$

Using the Shehu integral transform, one obtains the following:

$$\begin{aligned}
&S\{{}^C D_{0,t}^{\alpha} f(t)\} + S\{f(t)\} = \frac{2}{\Gamma(3-\alpha)} S\{t^{2-\alpha}\} + S\{t^2\},\\
&\left(\frac{s}{u}\right)^{\alpha} F(s,u) - \left(\frac{s}{u}\right)^{\alpha-1} f(0) + F(s,u) = \frac{2}{\Gamma(3-\alpha)} \Gamma(2-\alpha+1) \left(\frac{u}{s}\right)^{2-\alpha+1} + 2\left(\frac{u}{s}\right)^3,\\
&F(s,u)\left[1 + \left(\frac{s}{u}\right)^{\alpha}\right] = 2\left(\frac{u}{s}\right)^3 \left[1 + \left(\frac{u}{s}\right)^{\alpha}\right],\\
&F(s,u) = 2\left(\frac{u}{s}\right)^3.
\end{aligned}$$

Taking inverse Shehu transform, the exact analytical solution is obtained as:

$$f(t) = t^2. \tag{18}$$

Example 3 Consider the following linear fractional order in-homogeneous initial value problem with $\alpha \in (0,1)$:

$${}^C D_{0,t}^{\alpha} f(t) + f(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 - t, \quad f(0) = 0. \tag{19}$$

Using the Shehu integral transform, one obtains the following:

$$S\{{}^{C}D_{0,t}^{\alpha}f(t)\}+S\{f(t)\}=\frac{2}{\Gamma(3-\alpha)}S\{t^{2-\alpha}\}+\frac{1}{\Gamma(2-\alpha)}S\{t^{1-\alpha}\}+S\{t^{2}\}-S\{t\},$$

$$\left(\frac{s}{u}\right)^{\alpha}F(s,u)-\left(\frac{s}{u}\right)^{\alpha-1}f(0)+F(s,u)=\frac{2}{\Gamma(3-\alpha)}\Gamma(2-\alpha+1)\left(\frac{u}{s}\right)^{2-\alpha+1}$$

$$-\frac{1}{\Gamma(2-\alpha)}\Gamma(1-\alpha+1)\left(\frac{u}{s}\right)^{1-\alpha+1}+2\left(\frac{u}{s}\right)^{3}-\left(\frac{u}{s}\right)^{2},$$

$$F(s,u)\left[\left(\frac{s}{u}\right)^{\alpha}+1\right]=\left\{2\left(\frac{u}{s}\right)^{3}-\left(\frac{u}{s}\right)^{2}\right\}.\left[\left(\frac{u}{s}\right)^{-\alpha+1}\right],$$

$$F(s,u)=\left\{2\left(\frac{u}{s}\right)^{3}-\left(\frac{u}{s}\right)^{2}\right\}.$$

Taking the inverse Shehu transform, the exact analytical solution is obtained as:

$$f(t)=t^{2}-t. \tag{20}$$

Example 4 Consider the following linear fractional order in-homogeneous initial value problem with $\alpha\in[1,2]$:

$$D^{2}f(t)+{}^{C}D_{0,t}^{\alpha}f(t)+f(t)=1+t, \qquad f(0)=f'(0)=1. \tag{21}$$

Using the Shehu integral transform, one obtains the following:

$$S\{D^{2}f(t)\}+S\{{}^{C}D_{0,t}^{\alpha}f(t)\}+S\{f(t)\}=S\{1\}+S\{t\},$$

$$\left\{\left(\frac{s}{u}\right)^{2}F(s,u)-\left(\frac{s}{u}\right)f(0)-f'(0)\right\}+\left\{\left(\frac{s}{u}\right)^{\alpha}F(s,u)-\sum_{k=0}^{1}\left(\frac{s}{u}\right)^{\alpha-k-1}f^{(k)}(0)\right\}+F(s,u)$$

$$=\frac{u}{s}+\left(\frac{u}{s}\right)^{2},$$

$$\left\{\left(\frac{s}{u}\right)^{2}F(s,u)-\left(\frac{s}{u}\right)-1\right\}+\left\{\left(\frac{s}{u}\right)^{\alpha}F(s,u)-\left(\frac{s}{u}\right)^{\alpha-1}-\left(\frac{s}{u}\right)^{\alpha-2}\right\}+F(s,u)$$

$$=\frac{u}{s}+\left(\frac{u}{s}\right)^{2},$$

$$F(s,u)\left\{\left(\frac{s}{u}\right)^{\alpha}+\left(\frac{s}{u}\right)^{2}+1\right\}=\left(\frac{s}{u}\right)^{\alpha-2}+\left(\frac{s}{u}\right)^{\alpha-1}+\frac{s}{u}+1+\left(\frac{s}{u}\right)^{-1}\left(\frac{s}{u}\right)^{-2},$$

$$F(s,u)\left\{\left(\frac{s}{u}\right)^{\alpha}+\left(\frac{s}{u}\right)^{2}+1\right\}=\left(\left(\frac{s}{u}\right)^{\alpha}+\left(\frac{s}{u}\right)^{2}+1\right)\left(\left(\frac{u}{s}\right)^{2}+\frac{u}{s}\right),$$

$$F(s,u)=\left(\frac{u}{s}\right)^{2}+\frac{u}{s}.$$

Taking the inverse Shehu transform, the exact analytical solution is obtained as:

$$f(t)=t+1. \tag{22}$$

Hence, it is proved in this present section that the Shehu integral transform technique can now be used to solve any linear fractional order ordinary differential equation with prescribed condition (s) under the Caputo operator having fractional order $\alpha > 0$.

5. Conclusions

In this current research study, a new integral transform has been tested which is recently introduced with the name of Shehu integral transform and it is not so far used in the existing literature for the initial value problems of linear fractional order differential equations under the Caputo type operator. One of the contributions in the present work is the inclusion of a property called the Shehu transform for an integral of a function, found missing in the research work about the Shehu integral transform technique. Another achievement of this research work is a proof of a theorem which ultimately helps in finding the solution of fractional type of differential equations with the Caputo operator. Using this proposed theorem, different types of fractional order problems are analytically solved and can easily be compared with other existing transformation techniques. Thus, the Shehu integral transform is found to be useful not only for classical differential equations but also for fractional order differential equations which are defined under the Caputo differential operator. This research work may be extended to solve linear partial differential equations of fractional order with the Caputo operator via the Shehu integral transform technique.

References

[1] Peng, G. (2007). Synchronization of fractional order chaotic systems. *Physics Letters A*, *363*(5-6), 426-432.

[2] Atangana, A., & Qureshi, S. (2019). Modeling attractors of chaotic dynamical systems with fractal–fractional operators. *Chaos, Solitons & Fractals*, *123*, 320-337.

[3] Yusuf, A., Qureshi, S., Inc, M., Aliyu, A.I., Baleanu, D., & Shaikh, A.A. (2018). Two-strain epidemic model involving fractional derivative with Mittag-Leffler kernel. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *28*(12), 123121.

[4] Qureshi, S., & Yusuf, A. (2019). Modeling chickenpox disease with fractional derivatives: From Caputo to Atangana-Baleanu. *Chaos, Solitons & Fractals*, *122*, 111-118.

[5] Ullah, S., Khan, M.A., & Farooq, M. (2018). A fractional model for the dynamics of TB virus. *Chaos, Solitons & Fractals*, *116*, 63-71.

[6] Qureshi, S., & Yusuf, A. (2019). Fractional derivatives applied to MSEIR problems: Comparative study with real world data. *The European Physical Journal Plus*, *134*(4), 171.

[7] Qureshi, S., & Atangana, A. (2019). Mathematical analysis of dengue fever outbreak by novel fractional operators with field data. *Physica A: Statistical Mechanics and its Applications*, *526*, 121127.

[8] Khan, I. (2019). New idea of Atangana and Baleanu fractional derivatives to human blood flow in nanofluids. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *29*(1), 013121.

[9] Singh, J. (2019). A new analysis for fractional rumor spreading dynamical model in a social network with Mittag-Leffler law. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *29*(1), 013137.

[10] Singh, J., Kumar, D., Hammouch, Z., & Atangana, A. (2018). A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. *Applied Mathematics and Computation*, *316*, 504-515.

[11] Imran, M.A., Khan, I., Ahmad, M., Shah, N.A., & Nazar, M. (2017). Heat and mass transport of differential type fluid with non-integer order time-fractional Caputo derivatives. *Journal of Molecular Liquids*, *229*, 67-75.

[12] Qureshi, S., Yusuf, A., Shaikh, A.A., Inc, M., & Baleanu, D. (2019). Fractional modeling of blood ethanol concentration system with real data application. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *29*(1), 013143.

[13] Qureshi, S., Rangaig, N.A., & Baleanu, D. (2019). New numerical aspects of Caputo-Fabrizio fractional derivative operator. *Mathematics*, *7*(4), 374.

[14] Atangana, A., & Koca, I. (2016). Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order. *Chaos, Solitons & Fractals*, *89*, 447-454.

[15] Atangana, A. (2018). Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties. *Physica A: Statistical Mechanics and its Applications*, *505*, 688-706.

[16] Osman, M.S., Korkmaz, A., Rezazadeh, H., Mirzazadeh, M., Eslami, M., & Zhou, Q. (2018). The unified method for conformable time fractional Schrodinger equation with perturbation terms. *Chinese Journal of Physics*, *56*(5), 2500-2506.

[17] Atangana, A., & Gómez-Aguilar, J.F. (2018). Numerical approximation of Riemann-Liouville definition of fractional derivative: From Riemann-Liouville to Atangana-Baleanu. *Numerical Methods for Partial Differential Equations*, *34*(5), 1502-1523.

[18] Rezazadeh, H., Osman, M.S., Eslami, M., Mirzazadeh, M., Zhou, Q., Badri, S.A., & Korkmaz, A. (2019). Hyperbolic rational solutions to a variety of conformable fractional Boussinesq-Like equations. *Nonlinear Engineering*, *8*(1), 224-230.

[19] Maitama, S., & Zhao, W. (2019). New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. *International Journal of Analysis and Applications*, *17*(2), 167-190.

[20] Watugala, G.K. (1998). Sumudu transform-a new integral transform to solve differential equations and control engineering problems. *Mathematical Engineering in Industry*, *6*(4), 319-329.

[21] Srivastava, H.M., Minjie, L.U.O., & Raina, R.K. (2015). A new integral transform and its applications. *Acta Mathematica Scientia*, *35*(6), 1386-1400.

[22] Elzaki, T.M. (2011). The new integral transform 'Elzaki transform'. *Global Journal of Pure and Applied Mathematics*, *7*(1), 57-64.

[23] Xiao-Jun, Y.A.N.G. (2016). A new integral transform method for solving steady heat-transfer problem. *Thermal Science*, *20*(3), S639-S642.

[24] Belgacem, F.B.M., & Silambarasan, R. (2012). Theory of natural transform. *Journal - MESA*, *3*(1), 99-124.

[25] Podlubny, I. (1999). *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications* (Vol. 198). Elsevier.