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THE DETERMINANTS OF THE THREE-BAND BLOCK MATRICES

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Abstract. In the paper the method of calculating of the determinants of block matrices is presented. The three-band matrices are considered, both in the particular case (3D) as well as in the general case.

Introduction

When we attempt to solve many issues connected with heat flow or with the theory of vibrations the system of equations which reading to the band matrix are obtained. Typically, these systems are solved using the numerical methods. This work is an introduction to solve these systems using algebraic methods.

1. Solution of the problem

Let us consider the following matrices:

– the three-band matrix

$$A_1 = \begin{bmatrix} a & 1 & \dots & \dots & \dots & \dots & \dots \\ 1 & a & 1 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & 1 & a & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 & a \end{bmatrix}_{n_1 \times n_1} \tag{1}$$

and

– the three-band block matrix

$$A_k = \begin{bmatrix} A_{k-1} & I_{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_{k-1} & A_{k-1} & I_{k-1} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & I_{k-1} & A_{k-1} & I_{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & I_{k-1} & A_{k-1} \end{bmatrix}_{n_k \times n_k}, \quad k \geq 2 \tag{2}$$

So, the block matrices in 2D and in 3D case are presented in these form

$$A_2 = \begin{bmatrix} A_1 & I_1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_1 & A_1 & I_1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & I_1 & A_1 & I_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & I_1 & A_1 \end{bmatrix}_{n_2 \times n_2} \tag{3}$$

$$A_3 = \begin{bmatrix} A_2 & I_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_2 & A_2 & I_2 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & I_2 & A_2 & I_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & I_2 & A_2 \end{bmatrix}_{n_3 \times n_3} \tag{4}$$

etc.
It is easy to check that the determinant of the block matrix A_1 is as follows [1]

$$\det A_1 = a^{n_1} - \binom{n_1 - 1}{1} a^{n_1 - 2} + \binom{n_1 - 2}{2} a^{n_1 - 4} - \ldots \tag{5}$$

Then, following the same way we obtain the determinant of matrix A_2

$$\det A_2 = \det\left[A_1^{n_2} - \binom{n_2 - 1}{1} A_1^{n_2 - 2} + \binom{n_2 - 2}{2} A_1^{n_2 - 4} - \ldots \right] \tag{6}$$

Assuming the polynomial of degree n_2

$$\begin{aligned} f_2(x_1) &= x_1^{n_2} - \binom{n_2 - 1}{1} x_1^{n_2 - 2} + \binom{n_2 - 2}{2} x_1^{n_2 - 4} - \ldots = \\ &= (x_1 - p_{1,1})(x_1 - p_{1,2})(x_1 - p_{1,3}) \cdot \ldots \cdot (x_1 - p_{1,n_2}) \end{aligned} \tag{7}$$

where $p_{1,j}$ are zeros of the polynomial f, for $j = 1,\ldots,n_2$, we obtain the determinant of the three-band block matrix

$$\det A_2 = \det\left[\left(A_1 - p_{1,1}I_1\right)\left(A_1 - p_{1,2}I_1\right)\left(A_1 - p_{1,3}I_1\right)\cdot\ldots\cdot\left(A_1 - p_{1,n_1}I_1\right)\right] =$$
$$= \underbrace{\det\left(A_1 - p_{1,1}I_1\right)}_{W_{A_1}\left(p_{1,1}\right)}\cdot\underbrace{\det\left(A_1 - p_{1,2}I_1\right)}_{W_{A_1}\left(p_{1,2}\right)}\cdot\underbrace{\det\left(A_1 - p_{1,3}I_1\right)}_{W_{A_1}\left(p_{1,3}\right)}\cdot\ldots\cdot\underbrace{\det\left(A_1 - p_{1,n_1}I_1\right)}_{W_{A_1}\left(p_{1,n_1}\right)} =$$
$$= \left(\lambda_1 - p_{1,1}\right)\left(\lambda_1 - p_{1,2}\right)\left(\lambda_1 - p_{1,3}\right)\cdot\ldots\cdot\left(\lambda_1 - p_{1,n_1}\right)\cdot \tag{8}$$
$$\cdot\left(\lambda_2 - p_{1,1}\right)\left(\lambda_2 - p_{1,2}\right)\left(\lambda_2 - p_{1,3}\right)\cdot\ldots\cdot\left(\lambda_2 - p_{1,n_1}\right)\cdot$$
$$\cdot\ldots\cdot$$
$$\cdot\left(\lambda_{n_1} - p_{1,1}\right)\left(\lambda_{n_1} - p_{1,2}\right)\left(\lambda_{n_1} - p_{1,3}\right)\cdot\ldots\cdot\left(\lambda_{n_1} - p_{1,n_1}\right)$$

where $\lambda_1,\ldots,\lambda_{n_1}$ are eigenvalues of the matrix A_1 and $W_{A_1}(p_{1,1}),\ldots,W_{A_1}(p_{1,n_1})$ are the values of the characteristic polynomials W_{A_1} of the matrix A_1 [2, 3].
Moreover, the determinant of the matrix A_3 is given by the formula

$$\det A_3 = \det\left[A_2^{n_3} - \binom{n_3 - 1}{1}A_2^{n_3-2} + \binom{n_3 - 2}{2}A_2^{n_3-4} - \ldots\right] \tag{9}$$

and then applying the polynomial of degree n_3

$$f_2\left(x_2\right) = x_2^{n_3} - \binom{n_3 - 1}{1}x_2^{n_3-2} + \binom{n_3 - 2}{2}x_2^{n_3-4} - \ldots = \tag{10}$$
$$= \left(x_2 - p_{2,1}\right)\left(x_2 - p_{2,2}\right)\left(x_2 - p_{2,3}\right)\cdot\ldots\cdot\left(x_2 - p_{2,n_3}\right)$$

we obtain

$$\det A_3 = \det\left[\left(A_2 - p_{2,1}I_2\right)\left(A_2 - p_{2,2}I_2\right)\left(A_2 - p_{2,3}I_2\right)\cdot\ldots\cdot\left(A_2 - p_{2,n_2}I_2\right)\right] =$$
$$= \underbrace{\det\left(A_2 - p_{2,1}I_2\right)}_{W_{A_2}\left(p_{2,1}\right)}\cdot\underbrace{\det\left(A_2 - p_{2,2}I_2\right)}_{W_{A_2}\left(p_{2,2}\right)}\cdot\underbrace{\det\left(A_2 - p_{2,3}I_2\right)}_{W_{A_2}\left(p_{2,3}\right)}\cdot\ldots\cdot\underbrace{\det\left(A_2 - p_{2,n_2}I_2\right)}_{W_{A_2}\left(p_{2,n_2}\right)} =$$
$$= \underbrace{\det\left[A_1 - \left(p_{1,1} + p_{2,1}\right)I_1\right]}_{W_{A_1}\left(p_{1,1}+p_{2,1}\right)}\cdot\underbrace{\det\left[A_1 - \left(p_{1,1} + p_{2,2}\right)I_1\right]}_{W_{A_1}\left(p_{1,1}+p_{2,2}\right)}\cdot\ldots\cdot\underbrace{\det\left[A_1 - \left(p_{1,1} + p_{2,n_2}\right)I_1\right]}_{W_{A_1}\left(p_{1,1}+p_{2,n_2}\right)}\cdot$$
$$\cdot\ldots\cdot$$

$$\cdot\ldots\cdot$$

$$\cdot\underbrace{\det\left[A_1-\left(p_{1,n_1}+p_{2,1}\right)I_1\right]}_{W_{A_1}\left(p_{1,n_1}+p_{2,1}\right)}\cdot\underbrace{\det\left[A_1-\left(p_{1,n_1}+p_{2,2}\right)I_1\right]}_{W_{A_1}\left(p_{1,n_1}+p_{2,2}\right)}\cdot\ldots\cdot\underbrace{\det\left[A_1-\left(p_{1,n_1}+p_{2,n_2}\right)I_1\right]}_{W_{A_1}\left(p_{1,n_1}+p_{2,n_2}\right)} \tag{11}$$

Generally, continuing above algorithm we have

$$\det A_k=\det\left(A_{k-1}^{n_k}-r_{k-1,1}\,A_{k-1}^{n_k-2}+r_{k-1,2}\,A_{k-1}^{n_k-4}-\ldots\right) \tag{12}$$

Assuming

$$\det A_k=\det\left[A_{k-1}^{n_k}-\binom{n_k-1}{1}A_{k-1}^{n_k-2}+\binom{n_k-2}{2}A_{k-1}^{n_k-4}-\ldots\right] \tag{13}$$

and

$$\begin{aligned} f_k\left(x_{k-1}\right)&=x_{k-1}^{n_k}-\binom{n_k-1}{1}x_{k-1}^{n_k-2}+\binom{n_k-2}{2}x_{k-1}^{n_k-4}-\ldots=\\ &=\left(x_{k-1}-p_{k-1,1}\right)\left(x_{k-1}-p_{k-1,2}\right)\left(x_{k-1}-p_{k-1,3}\right)\cdot\ldots\cdot\left(x_{k-1}-p_{k-1,n_k}\right)\end{aligned} \tag{14}$$

we obtain

$$\det A_k=\underbrace{\det\left[A_1-\left(p_{1,1}+p_{2,1}+\ldots+p_{k-1,1}\right)I_1\right]}_{W_{A_1}\left(p_{1,1}+p_{2,1}+\ldots+p_{k-1,1}\right)}\cdot\ldots \tag{15}$$

The above expression includes $n_1\times n_2\times\ldots\times n_k$ factors.
In conclusion, the determinant of the three-band block matrix is given by the product of the respective characteristic polynomials of matrix A_1.
The determinants of the block matrices are applicable among others in problems of heat transfer. For example, the Finite Differences Method (FDM) for the Fourier equation (in the case of the internal nodes) leads to the seven-band internal equations in each time step and then the matrix of the system equations is a three-band block matrix.

References

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