ON COMPATIBILITY OF SOME TANGENCY RELATIONS OF SETS

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Abstract. In this paper some problems connected with the compatibility of the tangency relations of sets in generalized metric spaces are considered. After introducing in the first part of the paper the basic notions related to the above, in the second part we will give certain sufficient conditions for the compatibility of the tangency relations of sets of some classes.

Introduction

Let E be an arbitray non-empty set and let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of family E_0 of all non-empty subsets of set E.

Let l_0 be a function defined by the formula:

$$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for} \quad x, y \in E. \tag{1}$$

By some assumptions relating function l, function l_0 defined by formula (1) will be a metric of set E. For this reason, pair (E, l) can be treated as a certain generalization of a metric space and we shall call it (see [1]) generalized metric space. Using (1) we may define in a generalized metric space (E, l), similarly as in a metric space, the following notions: sphere $S_l(p, r)$ and ball $K_l(p, r)$ with the centre at point p and radius r.

Let $S_l(p, r)_u$ (see [2]) denote the so-called u-neighbourhood of sphere $S_l(p, r)$ in space (E, l).

Let k be any fixed positive real number and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 so that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0. \tag{2}$$

We say that pair (A, B) of sets $A, B \in E_0$ is (a, b)-clustered at point p of space (E, l), if 0 is a cluster point of the set of all real numbers $r > 0$ so that sets $A \cap S_l(p, r)_{a(r)}$ and $B \cap S_l(p, r)_{b(r)}$ are non-empty.

In paper [1] (see also [3, 4]) W. Waliszewski introduced in generalized metric space (E, l) the following definition of the tangency of sets:

$$T_l(a, b, k, p) = \{(A, B)\colon A, B \in E_0, \text{ pair of sets } (A, B) \text{ is } (a, b)\text{-clustered}$$
$$\text{at point } p \text{ of space } (E, l) \text{ and}$$
$$\frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0\}. \tag{3}$$

If $(A, B) \in T_l(a, b, k, p)$, then we say that set $A \in E_0$ is (a, b)-tangent of order k to set $B \in E_0$ at point p of space (E, l).

Set $T_l(a, b, k, p)$ defined by formula (4) we call (a, b)-tangency relation of order k at point $p \in E$ (or for short: tangency relation) of the sets in generalized metric space (E, l).

Definition 1. *Tangency relations of sets* $T_{l_1}(a_1, b_1, k, p)$, $T_{l_2}(a_2, b_2, k, p)$ *we call compatible in set* E *if*

$$(A, B) \in T_{l_1}(a_1, b_1, k, p) \;\Leftrightarrow\; (A, B) \in T_{l_2}(a_2, b_2, k, p) \;\; for \;\; A, B \in E_0. \tag{4}$$

Let ρ be an arbitrary metric of set E.

We say that set $A \in E_0$ has a Darboux property at point p of metric space (E, ρ), which we write as $A \in D_p(E, \rho)$ (see [2, 5]), if a number $\tau > 0$ exists so that set $A \cap S_\rho(p, r)$ is non-empty for $r \in (0, \tau)$.

Let $d_\rho A$ denote the diameter of set A and $\rho(A, B)$ the distance of sets A, B in metric space (E, ρ), i.e.

$$d_\rho A = \sup\{\rho(x, y) : x, y \in A\}, \quad \rho(A, B) = \inf\{\rho(x, y) : x \in A, \; y \in B\} \tag{5}$$

for $A, B \in E_0$.
By $\mathfrak{F}_\rho$ we will denote the class of all real function l fulfilling the conditions:

1^0 $\; l : E_0 \times E_0 \longrightarrow [0, \infty)$,

2^0 $\; \rho(A, B) \le l(A, B) \le d_\rho(A \cup B) \quad \text{for} \quad A, B \in E_0$.

From condition 2^0, it follows that

$$\rho(x, y) = \rho(\{x\}, \{y\}) \le l(\{x\}, \{y\}) \le d_\rho(\{x\} \cup \{y\}) = \rho(x, y),$$

whence we get equality

$$l(\{x\}, \{y\}) = \rho(x, y) \quad \text{for} \;\; x, y \in E. \tag{6}$$

Let f be any subadditive increasing and continous real function defined in a certain right-hand side neighbourhood of 0 so that $f(0) = 0$.

Let from the definition:

$$L(A,B) = f(l(A,B)) \quad \text{for } A,B \in E_0. \tag{7}$$

The class of all functions L defined by formula (7) we denote by $\mathfrak{F}_{f,\rho}$.
Hence and from 1^0 and 2^0 it follows that any function $L \in \mathfrak{F}_{f,\rho}$ fulfils the conditions:

3^0 $L : E_0 \times E_0 \longrightarrow [0, \infty)$,

4^0 $f(\rho(A,B)) \le L(A,B) \le f(d_\rho(A \cup B)) \quad \text{for} \quad A,B \in E_0$.

Because

$$f(\rho(x,y)) = f(\rho(\{x\},\{y\})) \le L(\{x\},\{y\}) \le f(d_\rho(\{x\} \cup \{y\})) = f(\rho(x,y)),$$

then from here it follows that

$$L(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for} \quad l \in \mathfrak{F}_{f,\rho} \quad \text{and} \quad x,y \in E. \tag{8}$$

We shall denote

$$\rho'(x,y) = f(\rho(x,y)) \quad \text{for} \quad x,y \in E. \tag{9}$$

Using the properties of function f, it is easy to check that function ρ' defined by formula (9) is the metric of set E. Hence it follows that any function $L \in \mathfrak{F}_{f,\rho}$ generates on set E metric ρ'.

If $f = \mathrm{id}$, where id denotes the identical function defined in a certain right-hand side neighbourhood of 0, then from the definition of functions l and L it follows that

$$L(A,B) = l(A,B) \quad \text{for } A,B \in E_0. \tag{10}$$

Let A' denote the set of all cluster points of set $A \in E_0$ and let

$$\rho(x,A) = \inf\{\rho(x,y) : \ y \in A\} \quad \text{for} \ \ x \in E. \tag{11}$$

Let us accept from the definition (see [2, 6])

$$\begin{gathered}\widetilde{M}_{p,k} = \{A \in E_0 : \ p \in A' \text{ and there exists } \mu > 0 \text{ such that} \\ \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \text{for every pair of points } (x,y) \in [A,p;\mu,k] \\ \text{if } \rho(p,x) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon\}\end{gathered} \tag{12}$$

where

$$[A,p;\mu,k] = \{(x,y)\colon x \in E,\ y \in A \text{ and } \mu\rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\}. \quad (13)$$

In Section 2 of this paper we will give certain sufficient conditions for compatibility of the tangency relations of sets of classes $\widetilde{M}_{p,k}$ having a Darboux property in metric space (E,ρ).

1. Compatibility of tangency relations of sets

Let ρ' be the metric of set E defined by formula (9). In paper [7] the following lemma was presented:

Lemma 1.1. *If $A \in D_p(E,\rho')$, then $A \in D_p(E,\rho)$ for any set $A \in E_0$.*

Because (see Lemma 1.1 of paper [5])

$$A \in D_p(E,\rho) \Rightarrow A \in D_p(E,\rho') \text{ for } A \in E_0 \quad (14)$$

then from Lemma 1.1 we get

Corollary 1.1. *If ρ' is the metric of set E defined by formula* (9), *then $A \in D_p(E,\rho)$ if and only if $A \in D_p(E,\rho')$ for any set $A \in E_0$.*

We shall denote by $d_{\rho'}A$ the diameter of set A, and by $\rho'(A,B)$ the distance of sets A,B in metric space (E,ρ').
In paper [7] I proved the following theorems:

Theorem 1.1. *If sets $A,B \in D_p(E,\rho)$ and functions a,b,f fulfil the conditions*:

$$a(f(r)) \le f(a(r)) \text{ and } b(f(r)) \le f(b(r)) \text{ for } r > 0 \quad (15)$$

$$f(r_1r_2) \le f(r_1)f(r_2) \text{ for } r_1,r_2 > 0 \quad (16)$$

then

$$(A,B) \in T_{d_\rho}(a,b,k,p) \Rightarrow (A,B) \in T_{d_{\rho'}}(a,b,k,p) \quad (17)$$

and

Theorem 1.2. *If sets $A,B \in D_p(E,\rho)$ and functins a,b,f fulfil inequalities* (15) *and*

$$f(r_1r_2) = f(r_1)f(r_2) \text{ for } r_1,r_2 > 0 \quad (18)$$

then

$$(A,B) \in T_{\rho'}(a,b,k,p) \Rightarrow (A,B) \in T_\rho(a,b,k,p). \quad (19)$$

From Theorems 5.1 and 5.4 of my monographic paper [6] (see also [2]) follows:

Corollary 1.2. *If functions* a_i, b_i $(i = 1, 2)$ *fulfil condition*

$$\frac{a_i(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \textit{and} \quad \frac{b_i(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{20}$$

then for arbitrary functions $L_1, L_2 \in \mathfrak{F}_{f,\rho}$, *tangency relations* $T_{L_1}(a_1, b_1, k, p)$ *and* $T_{L_2}(a_2, b_2, k, p)$ *are compatible in the classes of sets* $\widetilde{M}_{p,k} \cap D_p(E, \rho)$.

Hence, from Theorem 5.1 of this paper follows:

Corollary 1.3. *If sets* $A, B \in \widetilde{M}_{p,k} \cap D_p(E, \rho)$, *functions* a, b *fulfil condition*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \textit{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{21}$$

then for any function $L \in \mathfrak{F}_{f,\rho}$,

$$(A, B) \in T_{\rho'}(a, b, k, p) \Leftrightarrow (A, B) \in T_L(a, b, k, p) \Leftrightarrow (A, B) \in T_{d_{\rho'}}(a, b, k, p). \tag{22}$$

If $f = \mathrm{id}$, then from this corollary and from equality (9) immediately follows:

Remark 1.1. *For any function* $l \in \mathfrak{F}_\rho$,

$$(A, B) \in T_\rho(a, b, k, p) \Leftrightarrow (A, B) \in T_l(a, b, k, p) \Leftrightarrow (A, B) \in T_{d_\rho}(a, b, k, p) \tag{23}$$

when $A, B \in \widetilde{M}_{p,k} \cap D_p(E, \rho)$ *and functions* a, b *fulfil condition* (21).

If functions a, b, f fulfil conditions (15), (18), (21), $A, B \in \widetilde{M}_{p,k} \cap D_p(E, \rho)$, then from Theorems 1.1, 1.2 of this paper and from conditions (22), (23) it results the following diagram:

$$\begin{array}{ccc} (A, B) \in T_\rho(a, b, k, p) \Leftrightarrow (A, B) \in T_l(a, b, k, p) \Leftrightarrow (A, B) \in T_{d_\rho}(a, b, k, p) \\ \Uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow \\ (A, B) \in T_{\rho'}(a, b, k, p) \Leftrightarrow (A, B) \in T_L(a, b, k, p) \Leftrightarrow (A, B) \in T_{d_{\rho'}}(a, b, k, p) \end{array} \tag{24}$$

From this diagram follows:

Corollary 1.4. *If functions* a, b, f *fulfil conditions* (21), (15) *and* (18), *then tangency relations* $T_l(a, b, k, p)$ *and* $T_L(a, b, k, p)$ *are compatible in the classes of sets* $\widetilde{M}_{p,k} \cap D_p(E, \rho)$, *i.e.*

$$(A, B) \in T_l(a, b, k, p) \;\Leftrightarrow\; (A, B) \in T_L(a, b, k, p) \tag{25}$$

for $A, B \in \widetilde{M}_{p,k} \cap D_p(E, \rho)$.

From (24) and from Theorems 5.1, 5.4 of paper [6] (see Corollary 1.2) we also get:

Corollary 1.5. *If functions* a_i, b_i, f $(i = 1,2)$ *fulfil conditions* (20), (18) *and*

$$a_i(f(r)) \le f(a_i(r)) \quad \text{and} \quad b_i(f(r)) \le f(b_i(r)) \quad \text{for} \quad r > 0 \tag{26}$$

then tangency relations $T_l(a_1, b_1, k, p)$ *and* $T_L(a_2, b_2, k, p)$ *are compatible, i.e.*

$$(A,B) \in T_l(a_1, b_1, k, p) \;\Leftrightarrow\; (A,B) \in T_L(a_2, b_2, k, p) \tag{27}$$

for $A, B \in \widetilde{M}_{p,k} \cap D_p(E,\rho)$.

Moreover using diagram (24) and certain theorems from my monographic paper [6], we shall prove the following theorem:

Theorem 1.3. *If functions* a, b, f *fulfil conditions* (15) *and* (18)*, then for* $l_1, l_2 \in \mathfrak{F}_\rho$*, tangency relations* $T_{l_1}(a,b,k,p)$*,* $T_{l_2}(a,b,k,p)$ *are compatible in the classes of sets* $\widetilde{M}_{p,k} \cap D_p(E,\rho)$ *if and only if tangency relations* $T_{L_1}(a,b,k,p)$*,* $T_{L_2}(a,b,k,p)$ *for* $L_1, L_2 \in \mathfrak{F}_{f,\rho}$ *are compatible in these classes of sets.*

`Proof.` We assume that tangency relations $T_{l_1}(a,b,k,p)$, $T_{l_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho)$ for $l_1, l_2 \in \mathfrak{F}_\rho$, and let $(A,B) \in T_{l_1}(a,b,k,p)$ for $A, B \in \widetilde{M}_{p,k} \cap D_p(E,\rho)$. Hence, from the fact that function $d_\rho \in \mathfrak{F}_\rho$, we get

$$(A,B) \in T_{d_\rho}(a,b,k,p) \;\text{ dla }\; A, B \in \widetilde{M}_{p,k} \cap D_p(E,\rho), \tag{28}$$

which means that

$$\frac{1}{r^k} d_\rho((A \cap S_\rho(p,r)_{a(r)}) \cup (B \cap S_\rho(p,r)_{b(r)})) \xrightarrow[r \to 0^+]{} 0. \tag{29}$$

Because

$$d_\rho(A \cap S_\rho(p,r)_{a(r)}) \le d_\rho((A \cap S_\rho(p,r)_{a(r)}) \cup (B \cap S_\rho(p,r)_{b(r)}))$$

and

$$d_\rho(B \cap S_\rho(p,r)_{b(r)}) \le d_\rho((A \cap S_\rho(p,r)_{a(r)}) \cup (B \cap S_\rho(p,r)_{b(r)})),$$

then from (29), it follows that

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p,r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{30}$$

and

$$\frac{1}{r^k} d_\rho(B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0. \tag{31}$$

From conditions (30), (31) and from Lemma 3.2 of my monographic paper [6], it follows that functions a, b fulfil condition (21). Hence, from the assumptions of this theorem, from diagram (24) and Corollary 1.1 it results that

$$(A,B) \in T_{L_1}(a,b,k,p) \;\text{ for }\; A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho) \text{ and } L_1 \in \mathfrak{F}_{f,\rho}. \tag{32}$$

On the basis of Theorem 5.1 of paper [6], this condition is equivalent to condition

$$(A,B) \in T_{L_2}(a,b,k,p) \;\text{ for }\; A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho) \text{ and } L_2 \in \mathfrak{F}_{f,\rho},$$

which means the compatibility of tangency relations $T_{L_1}(a,b,k,p)$, $T_{L_2}(a,b,k,p)$ in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho)$.

Now we assume that tangency relations $T_{L_1}(a,b,k,p)$, $T_{L_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho)$ for $L_1, L_2 \in \mathfrak{F}_{f,\rho}$, and let

$$(A,B) \in T_{L_1}(a,b,k,p) \text{ for } A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho). \tag{33}$$

Hence and from Theorem 3.2 of my monographic paper [6], it follows that functions a, b fulfil condition (21). Because function $\rho' \in \mathfrak{F}_{f,\rho}$, then from here and from condition (33) we get

$$(A,B) \in T_{\rho'}(a,b,k,p) \text{ for } A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho). \tag{34}$$

Therefore, from Lemma 1.1 (see also Corollary 1.1), from condition (21), from the assumptions of this theorem and diagram (24), it follows that

$$(A,B) \in T_{l_1}(a,b,k,p) \text{ for } A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho) \text{ and } l_1 \in \mathfrak{F}_{\rho}. \tag{35}$$

Accepting $f = \mathrm{id}$ in Theorem 5.1 of paper [6] (see Remark 1.1), condition (35) is equivalent to condition

$$(A,B) \in T_{l_2}(a,b,k,p) \text{ for } A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho) \text{ and } l_2 \in \mathfrak{F}_{\rho},$$

which means that the tangency relations $T_{l_1}(a,b,k,p)$, $T_{l_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho)$.

Similarly, using the above-mentioned relationship, diagram (24) and Theorem 5.4 of paper [6], we can show:

Theorem 1.4. *If functions f, a_i, b_i $(i = 1, 2)$ fulfil conditions* (18) *and* (26)*, then for $l \in \mathfrak{F}_\rho$, tangency relations $T_l(a_1, b_1, k, p)$, $T_l(a_2, b_2, k, p)$ are compatible in the classes of sets $\widetilde{M_{p,k}} \cap D_p(E, \rho)$ if and only if tangency relations $T_L(a_1, b_1, k, p)$, $T_L(a_2, b_2, k, p)$ for $L \in \mathfrak{F}_{f,\rho}$ are compatible in these classes of sets.*

From Theorem 1.3 and 1.4 follows immediately the following corollary:

Corollary 1.6. *If functions f, a_i, b_i $(i = 1, 2)$ fulfil conditions* (18) *and* (26)*, then for $l_1, l_2 \in \mathfrak{F}_\rho$, tangency relations $T_{l_1}(a_1, b_1, k, p)$, $T_{l_2}(a_2, b_2, k, p)$ are compatible in the classes of sets $\widetilde{M_{p,k}} \cap D_p(E, \rho)$ if and only if tangency relations $T_{L_1}(a_1, b_1, k, p)$, $T_{L_2}(a_2, b_2, k, p)$ for $L_1, L_2 \in \mathfrak{F}_{f,\rho}$ are compatible in these classes of sets.*

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