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VALUATION AND DISCRETE VALUATION RINGS

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Abstract. In this article we consider non-commutative valuation and discrete valuation rings. We give equivalent conditions for a ring to be a valuation and a discrete valuation ring.

Introduction

The theory of valuation rings first was connected only with commutative fields. The theory of valuations and valuation rings have their beginning from the early 20th century. The concepts of valuations of fields and valuation domains first were introduced in 1932 by W. Krull in his famous paper [1]. In this paper a valuation ring was defined as an integral domain whose ideals are totally ordered by inclusion, i.e. commutative uniserial domains. He also showed the connection between the concepts of valuation domains and valuation rings of fields.

However, there is also a non-commutative side of this theory. In the non-commutative case there are different generalizations of valuation rings. The first generalization for valuation rings of division rings was obtained by Schilling in [2], who introduced the class of invariant valuation rings and systematically studied them in [3]. If we consider the invariant valuation rings of division rings which were introduced by Schilling in [2], one obtains that any invariant valuation ring is a semihereditary ring. Hence, semihereditary rings can be considered as some generalizations of Prüfer domains for non-commutative rings. A particular example of invariant valuation rings are discrete valuation rings, which are, besides only fields and division rings, the simplest class of rings. Nevertheless, they play an important role in algebra, number theory and algebraic geometry.

Another generalization of non-commutative valuation rings was introduced and studied by N.I. Dubrovin in [4]. These rings were named Dubrovin valuation rings after him. In this non-commutative valuation theory, any Dubrovin valuation ring of a simple Artinian ring Q is exactly a local semihereditary order of Q. Therefore, semihereditary orders can be considered as the global theory of Dubrovin valuation rings. Dubrovin valuation rings have found a large number of applications. More information about these rings and semihereditary orders in simple Artinian rings can be found in book [5].

In this paper we present and shortly discuss most of the basic results for non--commutative invariant valuation rings and discrete valuation rings of division rings.

All the rings considered in this paper are assumed to be associative with $1 \neq 0$, and all the modules are assumed to be unital. We write $U(A)$ for the group of units of a ring A, and D^* the multiplicative group of a division ring D. We refer to [6] for general material on the theory of rings and modules.

1. Valuation rings of division rings

The basic notion which plays the main role for the valuation theory is a totally ordered group.

Definition 1. A group G (with operation written by +) is said to be **totally ordered** (or **linearly ordered**) if there is a binary order relation $\geq$ in G which satisfies the following axioms:

(i) either $\alpha \geq \beta$ or $\beta \geq \alpha$;

(ii) if $\alpha \geq \beta$ and $\beta \geq \alpha$ then $\alpha = \beta$;

(iii) if $\alpha \geq \beta$ and $\beta \geq \gamma$ then $\alpha \geq \gamma$;

(iv) if $\alpha \geq \beta$ then $\gamma + \alpha \geq \gamma + \beta$ and $\alpha + \gamma \geq \beta + \gamma$

for all $\alpha, \beta, \gamma \in G$.

If $\geq$ is an order relation in a group G we shall write $\alpha > \beta$ if $\alpha \geq \beta$ and $\alpha \neq \beta$, we shall also write $\alpha \leq \beta$ if $\beta \geq \alpha$ and $\alpha < \beta$ if $\beta > \alpha$.

In the non-commutative case there are different generalizations of a valuation ring. We consider the generalization which was first proposed in 1945 by Schilling [2], who extended the concept of a valuation on a field to that on a division ring.

Definition 2. [2] Let G be a totally ordered group (written additively) with order relation $\geq$. Add to G a special symbol $\propto$ such that $x + \propto = \propto + x = \propto$ for all $x \in G$. Let D be a division ring. A **valuation** on D is a surjective map

$$v: D \to G \cup \{\propto\}$$

which satisfies the following:

1) $v(x) \leq \propto$;

2) $v(x) = \propto$ if and only if $x = 0$;

3) $v(xy) = v(x) + v(y)$;

4) $v(x + y) \geq \min(v(x), v(y))$,

for any $x, y \in D$.

Note that if D is a field, then from condition 3) it follows immediately that D admits only valuations with Abelian groups G.

Remark 1. Let D be a division ring with valuation v and multiplicative group D^*. Denote

$$U = \{u \in D^*: v(u) = 0\}$$

If $u_1, u_2 \in U$ then $v(u_1u_2) = v(u_1) + v(u_2) = 0$ and $v(u_2u_1) = v(u_2) + v(u_1) = 0$, i.e. $u_1u_2, u_2u_1 \in U$. Let 1 be the identity of D. Then $v(1) = v(1^2) = v(1) + v(1)$ implies that $1 \in U$. If $u \in U$ then $0 = v(1) = v(uu^{-1}) = v(u) + v(u^{-1}) = v(u^{-1})$, i.e. $u^{-1} \in U$. Thus U is a subgroup of D^* which is called the **group of valuation units**. Let $x \in D^*$. Then $v(xux^{-1}) = v(x) + v(u) + v(x^{-1}) = v(x) + v(x^{-1}) = v(xx^{-1}) = 0$ for any $u \in U$. Thus, U is an invariant subgroup of D^* which is equal to Ker(v). Therefore $D^*/U \simeq G$.

Proposition 1. Let $(G, +, \geq)$ be a totally ordered group, and let

$$v: D \to G \cup \{\propto\}$$

be a valuation of a division ring D. Then

$$A = \{x \in D : v(x) \geq 0\}$$

is a subring of D.

Proof. Let $x, y \in A$, then $v(x), v(y) \geq 0$. Therefore $v(xy) = v(x) + v(y) \geq 0$ and $v(x + y) \geq \min(v(x), v(y)) \geq 0$, which means that $xy \in A$ and $x + y \in A$. Moreover,

$$v(-x) = v((-1)\, x) = v(-1) + v(x) = v(x) \geq 0$$

for any $x \in$ A.

Definition 3. A subring A of a division ring D is called an **invariant valuation ring** (or **valuation ring** for short) of D if there is a totally ordered group G and a valuation

$$v: D \to G \cup \{\propto\}$$

of D such that

$$A = \{x \in D : v(x) \geq 0\}.$$

Lemma 1. Let A be a valuation ring of a division ring D with respect to valuation v. Then $U = U(A)$, where $U(A)$ is the group of valuation units of D, and $U = \{x \in D : v(x) = 0\}$.

Proof. Suppose that $u \in U(A)$, then there is element $w \in U(A)$ such that $uw = 1$. Therefore $0 = v(uw) = v(u) + v(w)$. So $v(u) = v(w) = 0$, since $v(u) \geq 0$ and $v(w) \geq 0$.

Conversely, suppose $u \in D$ and $v(u) = 0$. Then $u^{-1} \in D^*$ and $v(u^{-1}) = -v(u) = 0$. Hence $u, u^{-1} \in A$, which means that $u \in U(A)$.

For any invariant valuation ring A associated to the valuation v we denote

$$M = \{x \in D \ : \ v(x) > 0\} \ = A \setminus U,$$

the set of all non-units of A.

Lemma 2. An invariant valuation ring A is a local ring with the unique maximal left (and maximal right) ideal M of A.

Proof. Let $x, y \in M$ and $a \in A$. Then

1) $v(x+y) = \min(v(x),v(y)) > 0$, that is, $x + y \in M$;

2) $v(xa) = v(x) + v(a) > 0$ and $v(ax) = v(a) + v(x) > 0$, that is, $ax, xa \in M$.

Thus, M is an ideal of A. Show that M is the maximal ideal of A. Suppose that I is an ideal of A such that $M \,\delta\, I \,\phi\, A$. Since $M = A \setminus U$, there is a unit $u \in I$ such that $v(u) = v(u^{-1}) = 0$ and $u^{-1} \in A$. Consequently, $1 = uu^{-1} \in I$. Thus, $I = A$, i.e. M is a maximal ideal of A. Since $M = A \setminus U$, M consists of all non-units of A, therefore A is a local ring, and M is the unique maximal ideal of A., by proposition 10.1.1 [6].

Lemma 3. [2] If A is the valuation ring of a division ring D with respect to valuation v on D then both A and M are invariant subsets of D^*, that is, $dAd^{-1} = A$ and $dMd^{-1} = M$ for any $d \in D^*$.

Proof. Suppose that that $dAd^{-1} \not\subset A$ for some $d \in D^*$. Then there is an element $x = dyd^{-1} \in dAd^{-1}$ with $y \in A$ and $x \notin A$. Therefore $v(x) < 0$ and $v(y) \geq 0$. On the other hand, $y = d^{-1}xd$, and so

$$v(y) = v(d^{-1}) + v(x) + v(d) < v(d^{-1}) + v(d) = v(1) = 0,$$

since G is a totally ordered group. This contradiction shows that $dAd^{-1} = A$ for any $d \in D^*$.

Suppose that $dMd^{-1} \not\subset M$ for some $d \in D^*$. Then there is an element $x = dyd^{-1} \in dMd^{-1}$ with $y \in M$ and $x \notin M$. Since $dMd^{-1} \subset dAd^{-1} = A$ and $A = M \cup U$, $x \in U$. By remark 1, U is an invariant subgroup of D^*, and so $y = d^{-1}xd \in U$. Consequently, $y \in M \cap U = \varnothing$. This contradiction shows that M is an invariant subset of D^*.

The following theorem gives the equivalent definition of a valuation ring which is similar to the valuation domains of fields.

Theorem 1. (O.F.G. Schilling [2]) Let A be a subring of a division ring D. Then the following are equivalent:

1. A is a valuation ring with respect to some valuation v on D.
2. A is an invariant subring of D^*, and for any element $x \in D^*$ either $x \in A$ or $x^{-1} \in A$.

Proof. $1 \Rightarrow 2$. A is an invariant subring, by lemma 3. Suppose $x \in D^*$ and $x \notin A$, which means that $v(x) < 0$. Then $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$, hence $v(x^{-1}) = -v(x) \geq 0$. Thus $x^{-1} \in A$.

2 $\Rightarrow$ 1. Suppose that A is an invariant subring of a division ring D^* with the group of units $U(A)$. Let $u \in U(A)$ and $d \in D^*$. Then $x = dud^{-1} \in A$ and $x^{-1} = d^{-1}u^{-1}d \in A$. Therefore $x, x^{-1} \in U(A)$, i.e. $U(A)$ is an invariant subgroup of D^*.

Let $M = A \setminus U(A)$. Show that this set is also invariant in D^*. Let $d \in D^*$. Assume that $dMd^{-1} \neq M$. This means that there exists an element $x = dyd^{-1} \in dMd^{-1}$ with $y \in M$ and $x \notin M$. Note that $x \in A$, since A is an invariant subring in D^*. Therefore $x \in U(A)$ and $y = d^{-1}xd \in U(A)$, since $U(A)$ is invariant in D^*. So $y \in M \cap U(A) = \varnothing$. This contradiction shows that M is invariant in D^*.

Since $U(A)$ is an invariant subgroup in D^*, we can consider the factor group $G = D^*/U(A)$ as an additive group and define the natural map v: $D \to G \cup \{\infty\}$ such that $v(d) = dU(A) = U(A)d$ for each $d \in D^*$ and $v(0) = \infty$. Obviously, $v(du) = v(ud)$ for all $u \in U(A)$ and $d \in D^*$. We set $v(u) = 0$ for any $u \in U(A)$. Then v is a surjective map with $\mathrm{Ker}(v) = U(A)$. We must only introduce the total order on G assuming that $v(x) \leq \infty$ for all $x \in D$. Let $a, b \in D^*$. By assumption, either $a^{-1}b \in A$ or $b^{-1}a \in A$. Suppose $a^{-1}b \in A$, then $a(a^{-1}b)a^{-1} = ba^{-1} \in A$, since A is an invariant ring in D^*. We use this fact to order group G. We set $v(a) > v(b)$ in the case $ab^{-1} \in M$ (and $b^{-1}a \in M$). In this way G turns out to be totally ordered. Show that v is a valuation of D with valuation ring A. Indeed,

1) $v(x) \leq \infty$;
2) $v(x) = \infty$ if and only if $x = 0$;
3) v is surjective;
4) $v(d) = 0$ if and only if $d \in U(A)$;
5) $v(ab) = v(a)v(b)$.
6) Let $a,b \in D^*$ and $a+b \neq 0$. Assume that $v(a) > v(b)$ in G. This means that $ab^{-1} \in M$ or $ab^{-1} \in U(A)$. In both cases $ab^{-1}+1 \in A$, since $1 = 1^{-1} \in A$. Since $(a+b)b^{-1} = ab^{-1} + 1 \in A$, $v(a+b) \geq v(b) = \min(v(a), v(b))$. If $a + b = 0$, then $v(a+b) = \infty$ and we also have $v(a + b) \geq v(b) = \min(v(a), v(b))$.

This theorem gives a possibility to introduce other kinds of generalizations for a valuation ring of a division ring.

Definition 4. A subring A of a division ring D is called a **total valuation ring** if for each $x \in D^*$ we have $x \in A$ or $x^{-1} \in A$.

Theorem 1 states that any invariant valuation ring is a total valuation ring, but not conversely. Note that in the case of integral domains these two notions for valuation rings are equivalent to the notion of a classical valuation domain.

Lemma 4. (O.F.G. Schilling [2]) Let A be the valuation ring of a division ring D with a valuation v, and $a,b \in A$. Then the following statements are equivalent:

1) $a = bc_1$ with $c_1 \in A$;
2) $a = c_2b$ with $c_2 \in A$;
3) $v(a) \geq v(b)$.

Proof. 1), 2) $\Rightarrow$ 3). Suppose $a = bc_1 = c_2b$ with $c_1, c_2 \in A$. Then $v(a) = v(b) + v(c_1) = v(c_2) + v(b) \geq v(b)$ by condition 4) of definition 1.

3) $\Rightarrow$ 1), 2). Suppose $v(a) \geq v(b)$ and $b \neq 0$. Then $v(ab^{-1}) \geq 0$ and $v(b^{-1}a) \geq 0$, that is, $ab^{-1} \in A$ and $b^{-1}a \in A$. Thus, $a = b(b^{-1}a) = (ab^{-1})b$.

Suppose $v(a) \geq v(b)$ and $b = 0$. Then $v(b) = \infty$ and so $v(a) = \infty$, hence $a = 0$. This means that a is again both a left and a right multiple of b.

The next proposition gives the basic properties of invariant valuation rings.

Proposition 2. Let A be an invariant valuation ring of a division ring D with a valuation v. Then

1. $aA \subseteq bA$ or $bA \subseteq aA$ for any $a,b \in A$.
2. Each ideal of A is two-sided, i.e. A is a duo ring. (Recall that a ring A is called a **left** (**right**) **duo ring** if every left (right) ideal is two-sided. A **duo ring** means both a left and a right duo ring).
3. A is a right and a left Ore domain. Therefore it has a left and right classical ring of fractions which is a division ring.
4. Any finitely generated ideal of A is principal.

Proof. 1. This follows immediately from lemma 4.

2. Suppose that I is a left ideal of A, that is, $AI \subseteq I$. Since $1 \in A$, $AI = I$. Let $x = \sum_{i=1}^{n} y_i a_i$ be an arbitrary element of the set IA, where $y_i \in I$, $a_i \in A$. Then $v(y_i a_i) = v(y_i) + v(a_i) \geq v(y_i)$. Consequently, by lemma 4, $y_i a_i = b_i y_i$ for some $b_i \in A$. Therefore $x = \sum_{i=1}^{n} b_i y_i \in A I = I$. Thus I is a right ideal.

3. Let $I = xA$, then $AI = AxA = xA$, since I is a two-sided ideal. Analogously, $Ax = AxA$. Therefore $Ax = xA$. This means that A satisfies the right and left Ore conditions. Since A is a domain (that is, a ring without divisors of zero), A has a left and right classical ring of fractions which is a division ring.

4. Let $I = a_1A + a_2A + \dots + a_nA$, where $a_i \in A$. Since A is a valuation ring then we can choose among the elements $a_1,a_2,\dots,a_n$ an element with a minimal value. Without loss of generality, we can consider that $v(a_i) \geq v(a_1)$ for all i. Then, by lemma 4, this means that $a_iA \subseteq a_1A$. So $I = a_1A$.

As immediate consequences of this proposition, we obtain the following.

Corollary. Any invariant valuation ring of a division ring D is semihereditary and a Bézout ring. (Recall that a ring A is called a right Bézout ring if any of its finitely generated ideal is principal.)

The following theorem gives the equivalent definitions of a (non-commutative) invariant valuation ring.

Theorem 2. Let A be a ring with a division ring of fractions D which is invariant in D. Then the following are equivalent:

1. A is an invariant valuation ring of some valuation v on D.
2. The set of right (left) principal ideals of A is linearly ordered by inclusion.
3. The set of all ideals of A is linearly ordered by inclusion.

Proof.
1 $\Rightarrow$ 2. Let $a, b \in A$ and $v(a) \geq v(b)$. Then from lemma 4 it follows that $a \in bA$ and $a \in Ab$. Therefore $aA \subseteq bA$ and $Aa \subseteq Ab$.
2 $\Rightarrow$ 3. Let I and J be right ideals of A. Suppose that I is not contained in J. Choose a nozero element $x \in I \setminus J$. Let y be any element of J. Since $x \notin J$, $x \notin yA$, and so $xA \not\subseteq yA$. Therefore, by assumption, $yA \subseteq xA \subseteq I$. It follows that $J \subseteq I$.
3 $\Rightarrow$1. By assumption, A is a domain which has a division ring of fractions D. Let $x \in D$ be a nonzero element. Then $x = ab^{-1}$ for some nonzero $a,b \in A$. Since A is a uniserial ring, $Aa \subseteq Ab$ or $Ab \subseteq Aa$. If $Aa \subseteq Ab$ then $a = rb$ for some $r \in A$. Then $x = ab^{-1} = rbb^{-1} = r \in A$. If $Ab \subseteq Aa$ then $b = sa$ with $s \in A$. Then $x^{-1} = ba^{-1} =$ $= saa^{-1} = s \in A$. Since A is invariant in D by assumption, A is a valuation ring, by theorem 1.

2. Non-commutative discrete valuation rings

Similar to the commutative case of a field, one can introduce the notion of a discrete valuation ring of a division ring.

Definition 3. A subring A of a division ring D is called the (noncommutative) **discrete valuation ring** if there is a (discrete) valuation $v: D \to \mathbf{Z}$ of D such that

$$A = \{x \in D \ : \ v(x) \geq 0\}$$

The main example of noncommutative discrete valuation rings is a skew power series ring $K[[x, \sigma]]$ with $xa = \sigma(a)x$ for any $a \in K$, where K is a field and $\sigma: K \to K$ is a nontrivial automorphism of K.

We formulate the basic properties of a discrete valuation ring in the following proposition.

Proposition 3. Let A be a (noncommutative) discrete valuation ring of division ring D with respect to a valuation v. Let t be a fixed element of A with $v(t) = 1$. Then

1. A is a local domain with the nonzero unique maximal ideal $M = \{x \in A : v(x) > 0\}$.
2. Any nonzero element $x \in A$ has the unique representation in the form $x = t^n u = wt^n$, for some $u,w \in U(A)$, and $n \in \mathbf{Z}$, $n \geq 0$. If D is a division ring of fractions of A then any element $y \in D^*$ has the form $y = t^n u = wt^n$ for some $u,w \in U(A)$, and $n \in \mathbf{Z}$.
3. Any one-sided ideal I of A is a two-sided ideal and has the form $I = t^n A =$ $= At^n$ for some $n \in \mathbf{Z}$, $n \geq 0$, i.e. A is a principal ideal ring (Recall that a ring A is called a **principal ideal ring** if each one-sided ideal of A is principal). In particular, $M = tA = At$, and $I = M^n = t^n A = At^n$.

4. $\bigcap_{i=1}^{\infty} M^i = 0$, where M is the unique maximal ideal of A.
5. A is a Noetherian uniserial ring.
6. A is a hereditary ring.

Proof. 1. Since a discrete valuation ring A is a particular case of a valuation ring, this statement follows from lemma 2.

2. Let t be a fixed element of A with $v(t) = 1$, and $x \in A$ with $v(x) = n \geq 0$. Then $t \in M$, and $v(xt^{-n}) = v(x) - n = 0 = v(t^{-n}x)$. Therefore from lemma 1 it follows that $xt^{-n} = u \in U(A)$ and $t^{-n}x = u_1 \in U(A)$. So $x = ut^n = t^n u_1$.

Let $y \in D^*$. Since D is the division ring of fractions of A, any element $y \in D^*$ can represented in the form $y = ab^{-1}$ with $a,b \in A$. Let $a = t^n u$ and $b = t^m w$ with $u,w \in U(A)$ and $n,m \geq 0$. Then $y = (t^n u)(t^m w) = t^{n-m}u_1 w_1 = u_2 w_2 t^{n-m}$, where $u_1 w_1$, $u_2 w_2 \in U(A)$ and $n - m \in \mathbf{Z}$.

3. Since A is a valuation ring, any one-sided ideal of A is two-sided. Let I be an ideal of A. Choose in I an element x with a minimal value $v(x) = n$ (if there are a few such elements we can arbitrarily choose one). Then $x = t^n u = wt^n$ with $u,w \in U(A)$. Therefore $t^n A \subseteq I$ and $At^n \subseteq I$. Let $y \in I$, then $y = t^m w$ with $m \geq n$. So $v(t^{-n}y) \geq 0$, hence $t^{-n}y \in A$ and $y \in t^n A$. Therefore $I = t^n A$. Analogously, $I = At^n$. In particular, since $t \in M$, $M = tA = At$, and $M^n = t^n A = At^n = I$.

4. Assume that $N = \bigcap_{i=1}^{\infty} M^i \neq 0$. Let x be a nonzero element of N with $v(x) = n \geq 0$. Then $x = t^n u \in M^n$ with $u \in U(A)$. Since $x \in N$, $x \in M^{n+1}$. Therefore $x = t^{n+1}w$ with $w \in U(A)$. So $t^n u = t^{n+1}w$. Since A is a domain, $u = tw \in M$. A contradiction. Thus $N = 0$.

5. This follows immediately from 3 and theorem 2.

6. This follows from the fact that A is a principal ideal domain and any principal ideal over a domain is free.

Together with definition 3 there are other equivalent definitions of a discrete valuation domain which are given in the following statement.

Proposition 4. The following statements for a ring A are equivalent.
(1) A is a (noncommuative) discrete valuation domain.
(2) A is a local ring with nonzero maximal ideal M of the form $M = tA = At$, where $t \in A$ is a non-nilpotent element, and $\bigcap_{i=1}^{\infty} M^i = 0$.

Proof. (1) $\Rightarrow$ (2). From proposition 3 it follows that A is a local ring with nonzero maximal ideal M of the form $M = tA = At$, where $t \in M$ with $v(t) = 1$. Since A is a domain, t is a non-nilpotent element.
(2) $\Rightarrow$ (1). Since $M = tA = At$, it is easy to show directly that $M^n = t^n A = At^n$. Show that any nonzero element $x \in A$ has a unique representation in the form $x = t^n u = wt^n$,

where $u,w \in U(A)$ and $n \geq 0$. Let $x \notin U(A)$, then $x \in M$. Since $\bigcap_{i=1}^{\infty} M^i = 0$, there exists $n \geq 1$ such that $x \in M^n$ but $x \notin M^{n+1}$. Then $x = t^n u$, where $u \notin M$. Therefore $u \in U(A)$. Analogously $x = wt^n$.

Ring A is a domain. Otherwise there are elements $x,y \in A$ such that $xy = 0$. Let $x = t^n u$, $y = t^m w$ and $ut^m = t^m u_1$ with $u,w, u_1 \in U(A)$. Then $xy = t^{n+m}u_1 w = 0$, and so $t^{n+m} = 0$, which is not the case, since t is a non-nilpotent element. A contradiction.

Show that A is a right and left Ore domain. Let x,y be nonzero elements of A. Suppose $x = t^n u$, $y = t^m w$, $ut^m = t^m u_1$ and $wt^n = t^n w_1$ with $u,w, u_1, w_1 \in U(A)$. Then $xy = t^n ut^m w = t^n t^m u_1 w = t^m t^n u_1 w = t^m ww^{-1} t^n u_1 w = yx_1$, where $x_1 = w^{-1} t^n u_1 w \in A$. Analogously, $yx = xy_1$, where $y_1 = u^{-1} t^m w_1 u$. This shows that A satisfies the right and the left Ore conditions. So A has a division ring of fractions D. Any element of D^* can be represented in the form $d = ab^{-1}$, where $a, b \in A$. If $a = t^n u$ and $b = t^m w$ with $u,w \in U(A)$ and $n,m \geq 0$, then $d = t^{n-m}\varepsilon$, where $n - m \in \mathbf{Z}$ and $\varepsilon \in U(A)$. If we set $v(d) = v(t^{n-m}\varepsilon) = n - m \in \mathbf{Z}$, we obtain a valuation of D^* with the discrete valuation ring A.

This finishes the proof of the proposition.

Proposition 5. The the following statements for a ring A are equivalent.

(1) A is a (noncommuative) discrete valuation domain.

(2) A is a local principal ideal domain which is not a division ring.

(3) A is a Noetherian local ring with a nonzero maximal ideal which is two-sided and principal.

(4) A is a right (left) Noetherian local ring with the nonzero maximal ideal M of the form $M = tA = At$ with a non-nilpotent element $t \in A$.

Proof. That statement (1) implies each of the other properties was proved above. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). Let A be a Noetherian local ring whose maximal ideal $M \neq 0$, and $M = tA = At$. Note that $M^n \neq M^{n+1}$ for any $n \geq 0$. Otherwise, by the Nakayama lemma, $M^n = 0$, and $t^n = 0$, which is not the case, since t is a non-nilpotent element.

We now prove that $\bigcap_{i=1}^{\infty} M^i = 0$. Otherwise there is a nonzero element $x \in \bigcap_{i=1}^{\infty} M^i$. Then $x = a_0 = a_1 t = a_2 t^2 = \ldots = a_n t^n = \ldots$ for suitable $a_i \in A$.

Every $a_i \notin U(A)$. Otherwise $a_i - a_{i+1}t \in U(A)$, and from $a_i t^i = a_{i+1}t^{i+1}$ it would follow that $t^i = 0$, that is, t is nilpotent, which is not the case. So we have the ascending chain of right principal ideals $a_1 A \subset a_2 A \subset \ldots$ which must be stabilized because A is Noetherian, i.e. there is a number $n > 0$ such that $a_n A = a_{n+1} A$. Then $a_{n+1} = a_n b$ and $a_n = a_{n+1} c$ for some $b,c \in A$. Hence $a_{n+1} = a_n b = a_{n+1} cb$, and $a_{n+1}(1 - cb) = 0$. Since $1 - cb \in U(A)$, $a_{n+1} = 0$. So $x = 0$. This contradiction shows that $\bigcap_{i=1}^{\infty} M^i = 0$. Now we can apply proposition 4.

Proposition 6. The following statements for a ring A are equivalent.
(1) A is a (noncommuative) discrete valuation domain.
(2) A is a Noetherian non-Artinian uniserial ring.
(3) A is a Noetherian valuation ring.

Proof. Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) were proved in proposition 3. Implication (2) $\Rightarrow$ (3) follows from theorem 2.

(2) $\Rightarrow$ (1). Let A be a uniserial Noetherian but a non-Artinian ring. Then the unique maximal ideal M of A is the Jacobson radical of A, $M \neq 0$ and M^n/M^{n+1} is a simple A-module. So we have a strictly descending chain of ideals

$$A \supset M \supset M^2 \supset \ldots \supset M^n \supset \qquad (1)$$

Hence M is nilpotent, otherwise, (1) is a composition series for M and so A is an Artinian ring, which is not the case. Choose an element $t \in M \setminus M^2$. Since A is uniserial, $M^2 \subset tA \subseteq M$. Hence $M = tA$, since M/M^2 is a simple A-module. Analogously, $M = At$. Now we have exactly case (4) of proposition 5.
(3) $\Rightarrow$ (1). Let A be a Noetherian valuation ring. Then any ideal of A is finitely generated, hence it is principal by proposition 1. Thus, A is local principal ideal domain which is not a division ring. Therefore we can apply proposition 5.

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