WEIGHTED RESIDUAL METHOD AS A TOOL OF FDM ALGORITHM CONSTRUCTION

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Abstract. Weighted Residual Method can be treated as a kind of "common root" of well known algorithms such as FDM, FEM, BEM, collocational methods etc. This point of view is completely motivated and it is possible to show that all of these methods are the special cases of WRM. In this paper the construction of typical FDM algorithm for non-linear parabolic equations (on the basis of WRM) is presented. To simplify the considerations, the 1D task is discussed.

1. Direct approach

We consider the one-dimensional non-linear Fourier's equation in the form

$$c(T)\frac{\partial T(x,t)}{\partial t} = \frac{1}{x^m}\frac{\partial}{\partial x}\left[x^m \lambda(T)\frac{\partial T(x,t)}{\partial x}\right], \qquad a < x < b \tag{1}$$

with the following boundary and initial conditions

$$\begin{aligned} x = a: &\qquad T(a,t) = T_a(t) \\ x = b: &\qquad -\lambda\frac{\partial T(b,t)}{\partial x} = q_b(t) \\ t = 0: &\qquad T(x,0) = T_0(x) \end{aligned} \tag{2}$$

Equation (1) describes the non-steady temperature field in the volume of plate, cylinder or sphere (for m = 0, 1, 2, respectively), $c(T)$, $\lambda(T)$ are the thermophysical parameters (volumetric specific heat and thermal conductivity), x, t are the spatial co-ordinate and time.

The problem formulated in this way will be solved on the basis of FDM. So, in the considered area the set of points x_i, i = 1, 2, ...,N creating the differential mesh Δ^h with step $h = x_{i+1} - x_i$ = const is distinguished

$$\Delta^h: \; a = x_1 < x_2 < \ldots < x_i < \ldots < x_N = b - h/2 \tag{3}$$

At the same time the lattice

$$\Delta^t:\ 0=t^0<t^1<t^2<\ldots<t^F \tag{4}$$

with step Δt is introduced.

The cartesian product $\Delta^{ht}=\Delta^h\times\Delta^t$ constitutes the time-spatial differential mesh in which the approximate solution is searched.

For lattice Δ^h a three-points star created by central node x_i and two neighboring ones is defined. The numerical approximation of operator div (λ grad T) will be realized on the basis of mean differential quotient in the form

$$\left[\frac{\partial T(x,t)}{\partial x}\right]_i^p=\frac{T\left(x_i+\dfrac{h}{2},\,t^p\right)-T\left(x_i-\dfrac{h}{2},\,t^p\right)}{h} \tag{5}$$

In the interior of the segment $\langle x_{i-1}, x_{i+1}\rangle$ one distinguishes two additional points $x_{i-1/2}=x_i-h/2$, $x_{i+1/2}=x_i+h/2$ and then

$$\begin{aligned} x^m\lambda(T)\left[\frac{\partial T(x,t)}{\partial x}\right]_{i+1/2}^p&=\left(x_i+\frac{h}{2}\right)^m\lambda_{i+1/2}^p\frac{T_{i+1}^p-T_i^p}{h}\\ x^m\lambda(T)\left[\frac{\partial T(x,t)}{\partial x}\right]_{i-1/2}^p&=\left(x_i-\frac{h}{2}\right)^m\lambda_{i-1/2}^p\frac{T_i^p-T_{i-1}^p}{h}\end{aligned} \tag{6}$$

where index p identifies a certain moment of time, $t^p\in\langle t^f, t^{f+1}\rangle$. As a rule $t^p=t^f$ (explicit differential scheme) or $t^p=t^{f+1}$ (implicit differential scheme).
Analogously

$$\frac{1}{x^m}\frac{\partial}{\partial x}\left[x^m\lambda(T)\,\frac{\partial T(x,t)}{\partial x}\right]_i^p=\frac{1}{x_i^m}\frac{1}{h}\left[\left(x^m\lambda\frac{\partial T}{\partial x}\right)_{i+1/2}^p-\left(x^m\lambda\frac{\partial T}{\partial x}\right)_{i-1/2}^p\right] \tag{7}$$

in other words

$$\frac{1}{x^m}\frac{\partial}{\partial x}\left[x^m\lambda\,\frac{\partial T}{\partial x}\right]_i^p=\frac{T_{i+1}^p-T_i^p}{R_{i+1}^p}\,\Phi_{i+1}+\frac{T_{i-1}^p-T_i^p}{R_{i-1}^p}\,\Phi_{i-1} \tag{8}$$

where

$$\Phi_{i+1}=\frac{1}{h}\left(\frac{x_i+0.5h}{x_i}\right)^m,\qquad \Phi_{i-1}=\frac{1}{h}\left(\frac{x_i-0.5h}{x_i}\right)^m \tag{9}$$

and

$$R_{i+1}^p=\frac{0.5h}{\lambda_i^p}+\frac{0.5h}{\lambda_{i+1}^p},\qquad R_{i-1}^p=\frac{0.5h}{\lambda_i^p}+\frac{0.5h}{\lambda_{i-1}^p} \tag{10}$$

where R are the thermal resistances between the nodes of star, at the same time it is assumed that

$$\lambda^p_{i+1/2} = \frac{2\lambda^p_{i+1}\lambda^p_i}{\lambda^p_i + \lambda^p_{i+1}}, \qquad \lambda^p_{i-1/2} = \frac{2\lambda^p_i \lambda^p_{i-1}}{\lambda^p_i + \lambda^p_{i-1}} \tag{11}$$

The left side of equation (1) is approximated as follows

$$\frac{\partial T(x,t)}{\partial t} = c\left(T_i^{\,p}\right)\frac{T_i^{\,f+1} - T_i^{\,f}}{\Delta t} = c_i^{\,p}\frac{T_i^{\,f+1} - T_i^{\,f}}{\Delta t} \tag{12}$$

For example in the case $p = f$ one obtains

$$T_i^{\,f+1} = T_i^{\,f} + \frac{\Delta t}{c_i^{\,f}}\left[\frac{T_{i+1}^{\,f} - T_i^{\,f}}{R^f_{i+1}}\,\Phi_{i+1} + \frac{T_{i-1}^{\,f} - T_i^{\,f}}{R^f_{i-1}}\,\Phi_{i-1}\right] \tag{13}$$

As it is well known the differential equation (13) doesn't assure the stability of numerical solution and it is necessary to find permissible interval of time for assumed discretization of geometrical area.

Equation (13) has an obvious physical interpretation because it is transformated energy balance for the layer $x \in \langle x_{i-1/2}, x_{i+1/2}\rangle$, $t \in \langle t^f, t^{f+1}\rangle$ [1, 2].

2. Weighted residual method

A like as in the previous chapter we will consider the boundary initial problem in the form

$$\begin{aligned} &L(u) = b \\ &u\big|_{\Gamma_I} = T_a \\ &u\big|_{\Gamma_{II}} = q_b \\ &u\big|_{t=0} = T_0 \end{aligned} \tag{14}$$

where L is a certain differential operator.

Let us introduce the definition of solution defect in the interior of considered area and its boundary

$$\begin{aligned} &R = L(u) - b \\ &R_I = u - T_a \\ &R_{II} = q - q_b \end{aligned} \tag{15}$$

If u^* is the exact solution of the problem (14) then all defects are equal to zero, whereas in the case of approximate solution u they are non-zero values. If the assumed function u fulfills exactly boundary conditions (this case will be considered in this paper) then $R_I = R_{II} = 0$.

Weighted Residual Method requires an introduction of tapering function w and it is assumed that an approximate solution fulfills the following criterion

$$\int_\Omega R\, w\, \mathrm{d}\Omega = \int_\Omega \left[L(u) - b\right] w\, \mathrm{d}\Omega = 0 \tag{16}$$

where Ω is the considered area (in the general case it is a space-time domain). Assuming that tapering function w is created in the form of combination of linearly independent functions w_j

$$w = \sum_{j=1}^{M} \beta_j\, w_j \tag{17}$$

one obtains

$$\int_\Omega R\, w_j\, \mathrm{d}\Omega = \int_\Omega \left[L(u) - b\right] w_j\, \mathrm{d}\Omega = 0, \qquad j = 1, 2, ..., M \tag{18}$$

It can be shown [2-5] that the adequate choice of functions w_j leads to the typical algorithms of numerical solutions of boundary-initial problems e.g. FDM, FEM, BEM, collocational method etc.

3. WRM for non-linear parabolic equations

Let us consider the energy equation (1) with conditions (2). The time-spatial differential mesh $\Delta^{h\,t}$ has been described in chapter 1. In this mesh the four-points star shown in Figure 1 is distinguished. This one is typical for explicit differential schemes (one can also consider the others cases of FDM).

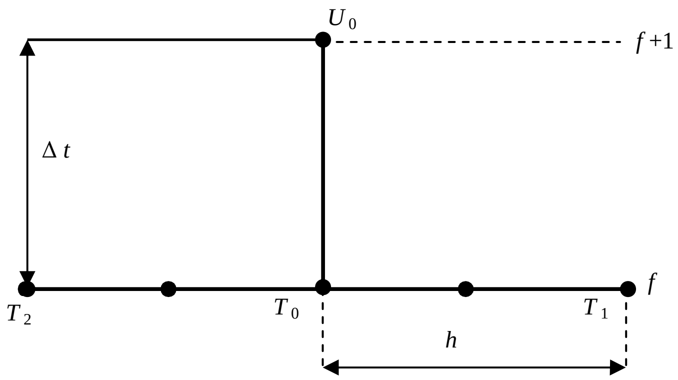

Fig. 1. Four-points star

The function $T(x, t)$ for $x \in \langle x_{i-1}, x_{i+1}\rangle$, $t \in \langle t^f, t^{f+1}\rangle$ is approximated by polynomial in the form

$$T = T_0 + \left(T_1 - T_2\right)\frac{x - x_0}{2h} + \left(T_1 - 2T_0 + T_2\right)\frac{\left(x - x_0\right)^2}{2h^2} + \left(U_0 - T_0\right)\frac{t - t^f}{\Delta t} \tag{19}$$

where

$$T_1 = T_{i+1}^{\,f}, \quad T_2 = T_{i-1}^{\,f}, \quad T_0 = T_i^{\,f}, \quad U_0 = T_i^{\,f+1}, \quad x_0 = x_i$$

One can easy to check that

$$\begin{gathered}\lambda x^m \left(\frac{\partial T}{\partial x}\right)_i^f = \lambda\left[\frac{T_1 - T_2}{2h}x^m + \frac{T_1 - 2T_0 + T_2}{h^2}\left(x^{m+1} - x^m x_0\right)\right] = \\ = \left(\lambda\frac{T_1 - T_0}{2h} - \lambda\frac{T_2 - T_0}{2h}\right)x^m + \left(\lambda\frac{T_1 - T_0}{h^2} + \lambda\frac{T_2 - T_0}{h^2}\right)\left(x^{m+1} - x^m x_0\right)\end{gathered} \tag{20}$$

It is assumed that the thermal conductivity λ concurrent the succeeding components in the last equation is averaged as in chapter 1 and then

$$\begin{gathered}\lambda x^m \left(\frac{\partial T}{\partial x}\right)_0^f = \frac{T_1 - T_0}{R_1}\left(\frac{x^{m+1} - x^m x_0}{h} + \frac{x^m}{2}\right) + \\ + \frac{T_2 - T_0}{R_2}\left(\frac{x^{m+1} - x^m x_0}{h} - \frac{x^m}{2}\right)\end{gathered} \tag{21}$$

or

$$\begin{gathered}\frac{1}{x^m}\frac{\partial}{\partial x}\left(\lambda x^m \frac{\partial T}{\partial x}\right) = \frac{T_1 - T_0}{R_1}\left[\frac{m+1}{h} + \frac{m}{x}\left(\frac{1}{2} - \frac{x_0}{h}\right)\right] + \\ + \frac{T_2 - T_0}{R_2}\left[\frac{m+1}{h} - \frac{m}{x}\left(\frac{1}{2} + \frac{x_0}{h}\right)\right]\end{gathered} \tag{22}$$

The function R in the area $x \in \langle x_0 - h, x_0 + h\rangle$, $t \in \langle t^f, t^{f+1}\rangle$ has the following form

$$\begin{gathered}R = \frac{T_1 - T_0}{R_1}\left[\frac{m+1}{h} + \frac{m}{x}\left(\frac{1}{2} - \frac{x_0}{h}\right)\right] + \\ + \frac{T_2 - T_0}{R_2}\left[\frac{m+1}{h} - \frac{m}{x}\left(\frac{1}{2} + \frac{x_0}{h}\right)\right] - c_0\frac{U_0 - T_0}{\Delta t}\end{gathered} \tag{23}$$

The tapering functions will be assumed as follows

$$w_i^f = 1, \quad (x, t) \in \Omega(x, t), \quad w_i^f = 0, \quad (x, t) \notin \Omega(x, t) \tag{24}$$

where

$$\Omega(x, t): \quad x_{i-1/2} \le x \le x_{i+1/2}, \quad t^f \le t \le t^{f+1} \tag{25}$$

The criterion of weighted residual method

$$\int_0^{t^F} \int_a^b R(x, t) w_i^f \, \mathrm{d}x \, \mathrm{d}t = 0, \qquad i = 2, \ldots, N-1, \qquad f = 0, 1, \ldots, F-1 \tag{26}$$

leads to the equations

$$\int_{t^f}^{t^{f+1}} \int_{x_i - h/2}^{x_i + h/2} R(x, t) \, \mathrm{d}x \, \mathrm{d}t = 0, \qquad i = 2, \ldots, N-1, \qquad f = 0, 1, \ldots, F-1 \tag{27}$$

The last formula can be written in the form

$$\begin{aligned} &\int_{t^f}^{t^{f+1}} \left[\int_{x_i - h/2}^{x_i} R(x, t) \, \mathrm{d}x + \int_{x_i}^{x_i + h/2} R(x, t) \, \mathrm{d}x \right] \mathrm{d}t = \\ &= \int_{t^f}^{t^{f+1}} \left[\int_{x_i}^{x_i + h/2} R(x, t) \, \mathrm{d}x - \int_{x_i}^{x_i - h/2} R(x, t) \, \mathrm{d}x \right] \mathrm{d}t = 0 \end{aligned} \tag{28}$$

Integrating from t^f to t^{f+1} and next from $x_i - h/2$ to $x_i + h/2$ one obtains

$$\begin{aligned} &\frac{T_1 - T_0}{R_1} \left\{ m + 1 + m \left(\frac{1}{2} - \frac{x_0}{h} \right) \left[\ln\left(1 + \frac{h}{2x_0} \right) - \ln\left(1 - \frac{h}{2x_0} \right) \right] \right\} + \\ &+ \frac{T_2 - T_0}{R_2} \left\{ m + 1 - m \left(\frac{1}{2} + \frac{x_0}{h} \right) \left[\ln\left(1 + \frac{h}{2x_0} \right) - \ln\left(1 - \frac{h}{2x_0} \right) \right] \right\} - c_0 \frac{U_0 - T_0}{\Delta t} h = 0 \end{aligned} \tag{29}$$

Because ln $(1+z) \approx z$ therefore one find the same result as in chapter 1. It can be shown that the similar considerations for node x_N lead to the equation presented in [2, 6].

Final remarks

The basic aim of the paper was to present more complex example of FDM algorithm construction on the basis of weighted residual method principles and it turned out that it is possible owning to proper choice of tapering functions and adequate approximation of function $T(x, t)$. It seems that the paper can be interested for the Readers working in the scope of numerical modelling of heat transfer processes.

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