EXPERIMENT DESIGN FOR ESTIMATION OF TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

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Abstract. The nonlinear Poisson equation is considered, in which the thermal conductivity is a function of temperature $\lambda(T) = p_1T+p_2$, where p_1, p_2 are the unknown parameters. To solve the inverse problem consisting in the identification of p_1 and p_2 the additional information connected with the knowledge of temperature T at the set of points (sensors) selected from the domain considered is necessary. The fundamental problem is the selection of sensors location and here the algorithm assuring the optimal sensors location is proposed. In the final part of the paper the results of computations are shown.

1. Formulation of the problem

The following 2D problem is considered

$$\begin{aligned} x\in\Omega:&\ \nabla\left[\lambda(T)\nabla T(x)\right]+Q(x)=0 \\ x\in\Gamma:&\ T(x)=T_b(x) \end{aligned} \tag{1}$$

where T is the temperature, $x = (x_1, x_2)$ are the spatial coordinates, $\lambda(T)$ is the thermal conductivity, $Q(x)$ is the source function, $T_b(x)$ is known boundary temperature. We assume that

$$\lambda(T) = p_1\, T + p_2 \tag{2}$$

where p_1, p_2 are the coefficients.

When the direct problem is considered then all geometrical and thermophysical parameters appearing in the mathematical model (1) are known.

In the paper the inverse parametric problem is discussed in which it is assumed that the coefficients p_1, p_2 are unknown. To solve the inverse problem the additional information is necessary. So, we assume that the temperatures at the selected points $x^i\in\Omega$ are given

$$T_{d_i} = T_d\,(x_1^i, x_2^i), \quad i=1,2,\ldots,N \tag{3}$$

where N is the number of sensors.

The accuracy of identification depends significantly on the choice of sensors location and this problem is here discussed.

2. Algorithm of optimal sensors location

Let $X = \{x^1, x^2,...,x^M\}$ denotes the set of spatial points at which measurements may be taken. The practical design problem consists in selection of corresponding weights $w_1, w_2,...,w_M$ which define the best experimental conditions [1]. To solve this problem the following iterative algorithm under the assumption that number of unknown parameters equals 2 and number of sensors equals N can be applied [2]. At first, the sensitivity matrix is constructed

$$\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ ... & ... \\ Z_{M1} & Z_{M2} \end{bmatrix} \tag{4}$$

where

$$Z_{l1} = \frac{\partial T(x^l, p_1^0, p_2^0)}{\partial p_1^0}, \quad Z_{l2} = \frac{\partial T(x^l, p_1^0, p_2^0)}{\partial p_2^0}, \; l = 1, 2, \ldots, M \tag{5}$$

and p_1^0, p_2^0 are the estimates of unknown parameters available e.g. from preliminary experiments.

Let

$$\mathbf{Z}(x^i) = [\, Z_{i1} \quad Z_{i2} \,] \tag{6}$$

and

$$\mathbf{S}(x^i) = \mathbf{Z}^{\mathbf{T}}(x^i)\,\mathbf{Z}(x^i) = \begin{bmatrix} Z_{i1}^2 & Z_{i1}Z_{i2} \\ Z_{i1}Z_{i2} & Z_{i2}^2 \end{bmatrix} \tag{7}$$

Step 1. Let $k = 0$ and *eps* is some positive tolerance. We assume that $S_0 = \{x^1, x^2,...,x^N\}$ denotes the initial sub-set of X.

Step 2. We set

$$w_l^k = \begin{cases} \dfrac{1}{M} & if \quad x^l \in S_k \\ 0 & if \quad x^l \in \mathrm{X} \setminus S_k \end{cases}, \; l = 1, 2, \ldots, M \tag{8}$$

Step 3. The following matrices are calculated

$$\mathbf{R}(\mathbf{w}^k) = \mathbf{Z}^{\mathbf{T}}\mathbf{w}^k\mathbf{Z} \tag{9}$$

this means

$$\mathbf{R}(\mathbf{w}^k)=\begin{bmatrix}\sum_{l=1}^{M} w_l^k Z_{l1}^2 & \sum_{l=1}^{M} w_l^k Z_{l1} Z_{l2}\\ \sum_{l=1}^{M} w_l^k Z_{l1} Z_{l2} & \sum_{l=1}^{M} w_l^k Z_{l2}^2\end{bmatrix} \tag{10}$$

and

$$\mathbf{P}(x^i,\mathbf{w}^k)=\mathbf{R}^{-1}(\mathbf{w}^k)\,\mathbf{S}(x^i) \tag{11}$$

where

$$\mathbf{w}^k=\begin{bmatrix} w_1^k & 0 & \dots & 0\\ 0 & w_2^k & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & w_M^k\end{bmatrix} \tag{12}$$

Step 4. Find

$$x^{*k}=\min_{x^i\in S_k}\left[\text{trace }\mathbf{P}(x^i,\mathbf{w}^k)\right] \tag{13}$$

and

$$x^{**k}=\max_{x^i\in X\setminus S_k}\left[\text{trace }\mathbf{P}(x^i,\mathrm{w}^k)\right] \tag{14}$$

Step 5. If

$$\text{trace }\mathbf{P}(x^{**k},\mathbf{w}^k)-\text{trace }\mathbf{P}(x^{*k},\mathbf{w}^k)>\text{eps} \tag{15}$$

then set

$$S_{k+1}=\left\{S_k\setminus\{x^{*k}\}\right\}\cup\{x^{**k}\} \tag{16}$$

increase k by one and go to *Step* 2, otherwise *Stop*.

It should be pointed out that the sensitivity coefficients (5) can be determined using the direct differentiation method [3, 4].

3. Sensitivity models

The equation (1) in Cartesian co-ordinate takes the form

$$\frac{\partial}{\partial x_1}\left[(p_1 T+p_1)\frac{\partial T(x_1,x_2)}{\partial x_1}\right]+\frac{\partial}{\partial x_2}\left[(p_1 T+p_2)\frac{\partial T(x_1,x_2)}{\partial x_2}\right]+\mathrm{Q}(x_1,x_2)=0 \tag{17}$$

or

$$(p_1 T+p_2)\left(\frac{\partial^2 T}{\partial x_1^2}+\frac{\partial^2 T}{\partial x_2^2}\right)+p_1\left(\frac{\partial T}{\partial x_1}\right)^2+p_1\left(\frac{\partial T}{\partial x_2}\right)^2+\mathrm{Q}=0 \tag{18}$$

At first, the equation (18) is differentiated with respect to the parameter p_1 and then

$$
\begin{gathered}
(T + p_1 Z_1)\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2}\right) + (p_1 T + p_2)\left(\frac{\partial^2 Z_1}{\partial x_1^2} + \frac{\partial^2 Z_1}{\partial x_2^2}\right) + \\
\left(\frac{\partial T}{\partial x_1}\right)^2 + \left(\frac{\partial T}{\partial x_2}\right)^2 + 2 p_1 \frac{\partial T}{\partial x_1}\frac{\partial Z_1}{\partial x_1} + 2 p_1 \frac{\partial T}{\partial x_2}\frac{\partial Z_1}{\partial x_2} = 0
\end{gathered}
\tag{19}
$$

where $Z_1 = Z_1(x_1, x_2) = \partial T(x_1, x_2)/\partial p_1$.

In similar way the equation (18) is differentiated with respect to p_2, namely

$$
\begin{gathered}
(p_1 Z_2 + 1)\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2}\right) + (p_1 T + p_2)\left(\frac{\partial^2 Z_2}{\partial x_1^2} + \frac{\partial^2 Z_2}{\partial x_2^2}\right) + \\
2 p_1 \frac{\partial T}{\partial x_1}\frac{\partial Z_2}{\partial x_1} + 2 p_1 \frac{\partial T}{\partial x_2}\frac{\partial Z_2}{\partial x_2} = 0
\end{gathered}
\tag{20}
$$

where $Z_2 = Z_2(x_1, x_2) = \partial T(x_1, x_2)/\partial p_2$.
The sensitivity equations (19), (20) are supplemented by conditions

$$
(x_1, x_2) \in \Gamma: \quad Z_1(x_1, x_2) = 0, \quad Z_2(x_1, x_2) = 0 \tag{21}
$$

The basic problem (1) and sensitivity ones (19), (20), (21) for the assumed values of parameters p_1, p_2 are solved by means of the finite difference method [5, 6].
On the basis of these solutions the sensitivity matrix (4) can be constructed.

4. Results of computations

The square of dimensions 0.1 m×0.1 m is considered. On the right and upper surfaces the boundary temperature 100°C is assumed, on the remaining parts of the boundary the temperature 600°C has been accepted. In Figure 1 the mesh used in FDM is shown. Figures 2 and 3 illustrates the distribution of sensitivity functions $Z_1(x_1, x_2)$, $Z_2(x_1, x_2)$ under the assumption that $p_1 = -0.05$, $p_2 = 380$ and Q = 0.

In the case of parameters p_1, p_2 identification in order to determine the optimum sensors location the algorithm presented in chapter 2 is applied. The set X of spatial points at which the measurements have been taken is marked by 1, 2,...,100 in Figure 1 (M = 100). Under the assumption that number of sensors equals to N = 2 the optimal sensors position corresponds to the nodes 28 and 54, respectively, while for N = 3 the optimal sensors location corresponds to the nodes 28, 45 and 54. Using the algorithm it is possible to consider the greater number of sensors, but from the practical point of view the number of thermocouples should be rather small.

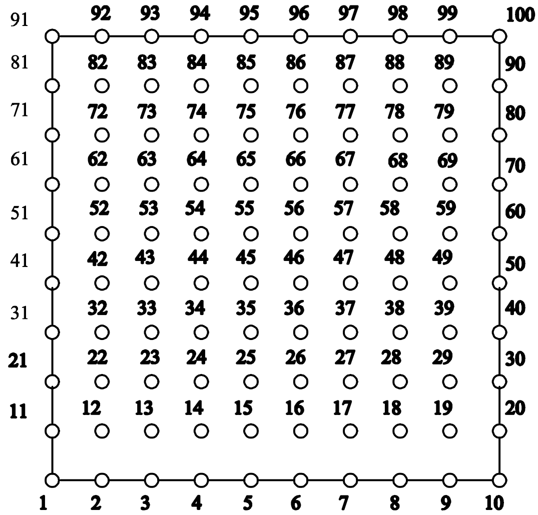

Fig. 1. The mesh - potential sensors locations

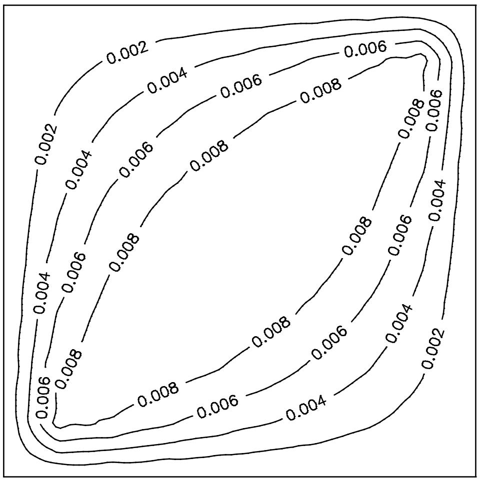

Fig. 2. Distribution of function Z_1

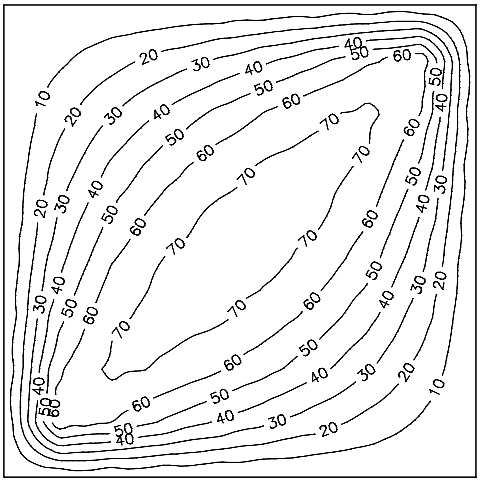

Fig. 3. Distribution of function Z_2

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