ON THE QUEUE-LENGTH DISTRIBUTION IN THE GI^X/G/1 SYSTEM WITH SERVER VACATIONS AND EXHAUSTIVE SERVICE

Wojciech M. Kempa

Institute of Mathematics, Silesian University of Technology
Wojciech.Kempa@polsl.pl

Abstract. In the article the $GI^X/G/1$ queueing system with server vacations and exhaustive service is investigated. For single and multiple vacations the queue-length transient distribution is studied first for a certain simplified system. Using the formula of total probability we direct the analysis to that for the system without vacations. The general case is obtained by applying the renewal theory approach. Explicit representations for Laplace transforms of queue-size distributions in systems with single and multiple vacations are obtained.

Introduction

In the article we consider the $GI^X/G/1$ queueing system with group arrivals and server vacations. The system works in the "standard" regime i.e. it starts working at time $t = 0$ when the first group enters and is empty before initializing. The service discipline is supposed to be of FIFO type. Every time when the system becomes empty, the server begins a vacation time in that the service process is stopped. We distinguish two types of such periods:

- single vacation, after that - if the system is empty - the server "waits" for the first arriving group and the service process begins immediately;
- multiple vacation, when the server begins successive vacation times as far as there are customers waiting in the queue after one of them.

The exact transient analysis of non-Markovian queueing systems is very difficult but possible. One can find the proposal of new approach and some results for the queue-length distribution, virtual waiting time and departure process in [1-4], where the $GI^X/G/1$ system without vacations is considered. The results for group arrival queueing systems with server vacations concern mainly systems with compound Poisson process as the arrival process (see [5-7] and [8]). The server vacation systems with Poisson input stream are widely described in [9] too.

In the paper we study the transient queue-length distribution for systems with single and multiple vacations. First, we consider a certain simplified system. Applying the formula of total probability we direct the analysis to that for the system without vacations under two different initial conditions: for the "standard" regime and with fixed number of customers present just after the opening. The general case is obtained using the renewal theory approach.

The article is organized as follows. In Section 2 we state necessary definitions and notations. Section 3 presents the case of the system with single vacations. The system with multiple vacations is investigated in Section 4. In Section 5 we prove some auxiliary formulae which complete results obtained in two previous sections.

1. Preliminaries

Let us suppose that interarrival times are independent and identically distributed (i.i.d.) random variables with a distribution function (d.f.) $F_1(\cdot)$, service times are i.i.d. random variables with a d.f. $F_2(\cdot)$ and number of customers in the arriving groups are distributed by the sequence $\{p_k\}$. As usual we assume mutually independence of interarrival times, service times and group sizes.

Let $\overline{F_i}(t) = 1 - F_i(t)$, $i = 1, 2$ and

$$f_i(s) = \int_0^\infty e^{-sx} dF_i(x), \quad s > 0,\ i = 1, 2, \quad p(\theta) = \sum_{k=1}^\infty p_k \theta^k, \ \ |\theta| \le 1. \qquad (1)$$

By $F_i^{i*}(\cdot)$ we denote the i-fold convolution of the d.f. $F_i(\cdot)$ with itself and by $\{p_k^{i*}\}$ - the i-fold convolution of the sequence $\{p_k\}$ with itself. Let $\widehat{\xi}(t)$ be the number of customers present at time t in the "ordinary" system (without server vacations). Let besides $\mathbf{P}_{\mathrm{std}}\{\cdot\}$ and $\mathbf{P}_n\{\cdot\}$ denote probabilities under two different initial conditions for the "ordinary" system: in the "standard" regime and with n customers present at time $t = 0+$ respectively. Denote by $\widehat{\tau}_1$ the first busy period of the "ordinary" system.

The system with server vacations we consider on successive vacation cycles c_i, $i = 0, 1, 2, ...$ - for the system with single vacations, and C_i, $i = 0, 1, 2, ...$ - for the system with multiple vacations. Since these systems work in the "standard" regime then $\mathbf{P}\{c_0 < t\} = \mathbf{P}_{\mathrm{std}}\{\widehat{\tau}_1 < t\}$ and

$$c_i = v_i + \delta_i + \tau_i, \quad i = 1, 2, ..., \qquad (2)$$

where v_i is a single vacation time that starts the cycle c_i, δ_i is the idle time (0 if there are arrivals during v_i) and τ_i is the busy period during c_i. Similarly, for the system with multiple vacations we have $\mathbf{P}\{C_0 < t\} = \mathbf{P}_{\mathrm{std}}\{\widehat{\tau}_1 < t\}$ and

$$C_i = \sum_{j=1}^{A_i} V_{ij} + T_i, \quad i = 1, 2, ..., \tag{3}$$

where T_i is the *i*-th busy period and A_i denotes the number of single vacation times V_{ij} contained in C_i. We assume that v_i and V_{ij} for $i, j \geq 1$ are i.i.d. random variables with a d.f. $G(\cdot)$ and are independent on the arrival process.

2. The number of customers ξ(t) present in the system with single vacations

Let us consider the system with single vacations v_i. We are interested in the explicit formula for the Laplace transform of probability function (p.f.) of $\xi(t)$ i.e. for the expression

$$\int_0^\infty e^{-\lambda t} \mathbf{P}\{\xi(t) = m\} dt, \quad m \geq 0, \ \lambda > 0. \tag{4}$$

Assume, as usual, that successive vacation cycles c_i are independent random variables. Denote by $B_0(\cdot)$ and $B_1(\cdot)$ d.fs of c_0 and c_i, $i = 1, 2, ...$ respectively. Let

$$\mathbf{P}_0\{\cdot\} = \mathbf{P}\{\cdot \mid c_0 = 0\}$$

denote the probability on condition that the system starts working empty at time $t = 0$ and the first vacation time (and, of course, vacation cycle c_1) begins at this time (the system "waits" for customers).

Since moments at which successive vacation cycles c_i, $i = 0, 1, 2, ...$ begin are renewal moments, then, defining the renewal function of delayed renewal process generated by random variables c_i, $i = 0, 1, ...$ as

$$\Phi(t) = \sum_{i=1}^\infty \left(B_0 * B_1^{(i-1)*}\right)(t), \tag{5}$$

we have for $m \geq 1$

$$\mathbf{P}\{\xi(t) = m\} = \sum_{i=0}^\infty \mathbf{P}\{\xi(t) = m, \, t \in c_i\} = \mathbf{P}_{\mathrm{std}}\{\widehat{\xi}(t) = m, \, t \in \widehat{\tau}_1\}$$
$$+ \sum_{i=1}^\infty \int_0^t \mathbf{P}_0\{\xi(t-y) = m, \, t-y \in c_1\} d\left(B_0 * B_1^{(i-1)*}\right)(y)$$
$$= \mathbf{P}_{\mathrm{std}}\{\widehat{\xi}(t) = m, \, t \in \widehat{\tau}_1\} + \int_0^t \mathbf{P}_0\{\xi(t-y) = m, \, t-y \in c_1\} d\Phi(y) \tag{6}$$

and besides

$$\mathbf{P}\{\xi(t)=0\}=\sum_{i=1}^{\infty}\mathbf{P}\{\xi(t)=0,\, t\in c_i\}=\int_0^t \overline{F}_1(t-y)d\Phi(y). \tag{7}$$

Now we will find the representation for the expression $\mathbf{P}_0\{\xi(t)=m,\, t\in c_1\}$. The formula of total probability leads to

$$\begin{aligned}
\mathbf{P}_0\{\xi(t)=m,\, t\in c_1\} &= \int_0^t dG(y)\int_y^t \mathbf{P}_{\rm std}\{\widehat{\xi}(t-z)=m,\, t-z\in\widehat{\tau}_1\}dF_1(z)\\
&+\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}p_n^{i*}\int_0^t\big[F_1^{i*}(y)-F_1^{(i+1)*}(y)\big]\mathbf{P}_n\{\widehat{\xi}(t-y)=m,\, t-y\in\widehat{\tau}_1\}dG(y)\\
&+\overline{G}(t)\sum_{i=1}^{m}p_m^{i*}\big[F_1^{i*}(t)-F_1^{(i+1)*}(t)\big],\quad m\ge 1.
\end{aligned} \tag{8}$$

Let us briefly comment the right side of (8).The first summand presents the situation in that on the first cycle c_1 the vacation time v_1 ends before time t and the first group arrives after finishing v_1, but still before t. Hence we can describe the state of the system at time t by means of the state of "ordinary" one working in "standard" regime. In the second summand of (8) v_1 ends before t and there are arrivals during the vacation time. Hence, the state at time t can be described by means of the state of "ordinary" system with fixed number of customers present just after the opening. The last summand on the right side of (8) describes the situation in that we have arrivals before t, but v_1 ends after t. Introducing the Laplace transform on the argument t we can rewrite (8) in the following form

$$\begin{aligned}
q(m,\lambda) &= \int_0^\infty e^{-\lambda t}\mathbf{P}_0\{\xi(t)=m,\, t\in c_1\}dt=\widehat{Q}_{\rm std}(m,\lambda)\int_0^\infty e^{-\lambda z}G(z)dF_1(z)\\
&+\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\widehat{Q}_n(m,\lambda)p_n^{i*}\int_0^\infty e^{-\lambda y}\big[F_1^{i*}(y)-F_1^{(i+1)*}(y)\big]dG(y)\\
&+\sum_{i=1}^{m}p_m^{i*}\int_0^\infty e^{-\lambda t}\overline{G}(t)\big[F_1^{i*}(t)-F_1^{(i+1)*}(t)\big]dt,\quad m\ge 1,
\end{aligned} \tag{9}$$

where

$$\widehat{Q}_{\rm std}(m,\lambda)=\int_0^\infty e^{-\lambda t}\mathbf{P}_{\rm Pstd}\{\widehat{\xi}(t)=m,\, t\in\widehat{\tau}_1\}dt, \tag{10}$$

$$\widehat{Q}_n(m,\lambda)=\int_0^\infty e^{-\lambda t}\mathbf{P}_n\{\widehat{\xi}(t)=m,\, t\in\widehat{\tau}_1\}dt. \tag{11}$$

Formulae (6), (7) and (9) allow to state the following theorem giving the representation for the Laplace transform of p.f. of the queue-size in the system with single vacation times.

Theorem 1. *For any* $\lambda > 0$ *we have*

$$\int_0^\infty e^{-\lambda t}\mathbf{P}\{\xi(t)=m\}dt = \widehat{Q}_{\mathrm{std}}(m,\lambda) + q(m,\lambda)\int_0^\infty e^{-\lambda y}d\Phi(y), \quad m \geq 1 \tag{12}$$

and

$$\int_0^\infty e^{-\lambda t}\mathbf{P}\{\xi(t)=0\}dt = \frac{1-f_1(\lambda)}{\lambda}\int_0^\infty e^{-\lambda y}d\Phi(y), \tag{13}$$

where $q(m,\lambda)$ *and* $\widehat{Q}_{\mathrm{std}}(m,\lambda)$ *are defined in (9) and (10) respectively.*

3. The number of customers ξ(t) present in the system with multiple vacations

Let us consider the system with multiple vacations. Of course C_0 has a d.f. $B_0(\cdot)$. Let $B_2(\cdot)$ be the d.f. of random variables C_i for $i = 1, 2, \ldots$. Denoting

$$\Psi(t) = \sum_{i=1}^\infty \big(B_0 * B_2^{(i-1)*}\big)(t) \tag{14}$$

we have

$$\mathbf{P}\{\xi(t)=0\} = \sum_{i=1}^\infty \mathbf{P}\{\xi(t)=0,\, t\in C_i\} = \int_0^t \overline{F}_1(t-y)d\Psi(y). \tag{15}$$

For $m \geq 1$ we obtain

$$\mathbf{P}\{\xi(t)=m\} = \sum_{i=0}^\infty \mathbf{P}\{\xi(t)=m,\, t\in C_i\} = \mathbf{P}_{\mathrm{std}}\{\widehat{\xi}(t)=m,\, t\in\widehat{\tau}_1\}$$
$$+\int_0^t \mathbf{P}_0\{\xi(t-y)=m,\, t-y\in C_1\}d\Psi(y). \tag{16}$$

The representation for $\mathbf{P}_0\{\xi(t)=m,\, t\in C_1\}$ we will find using the formula of total probability. We have for $m \geq 1$

$$\mathbf{P}_0\{\xi(t)=m,\, t\in C_1\}=\sum_{i=1}^{\infty}\int_0^t dF_1(x)\int_0^x dG^{(i-1)*}(y)\int_{x-y}^{t-y} dG(z)\times$$
$$\times\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} p_n^{(k+1)*}\big[F_1^{k*}(z-x+y)-F_1^{(k+1)*}(z-x+y)\big]\times$$
$$\times\,\mathbf{P}_n\{\widehat{\xi}(t-y-z)=m,\, t-y-z\in\widehat{\tau}_1\}$$
$$+\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} p_m^{(k+1)*}\int_0^t\big[F_1^{k*}(t-x)-F_1^{(k+1)*}(t-x)\big]\int_0^x dG^{(i-1)*}(y)\int_{t-y}^{\infty} dG(z)dF_1(x). \tag{17}$$

In the above formula the first summand concerns the situation in that the vacation time ends before t. The second summand describes the situation in that the first arrival occurs before t but the vacation time ends after t.

Introducing in (17) the Laplace transform on the argument t we get

$$Q(m,\lambda)=\int_0^{\infty} e^{-\lambda t}\mathbf{P}_0\{\xi(t)=m,\, t\in C_1\}dt$$
$$=\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} p_n^{(k+1)*}\widehat{Q}_n(m,\lambda)\int_0^{\infty} e^{-\lambda y}dG^{(i-1)*}(y)\int_0^{\infty} e^{-\lambda z}H(y,z,k)dG(z)$$
$$+\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} p_m^{(k+1)*}\int_0^{\infty} e^{-\lambda t}\int_0^t\big[F_1^{k*}(t-x)-F_1^{(k+1)*}(t-x)\big]B(t,x)dF_1(x)dt, \tag{18}$$

where for $k\geq 0$

$$H(y,z,k)=\int_y^{y+z}\big[F_1^{k*}(y+z-x)-F_1^{(k+1)*}(y+z-x)\big]dF_1(x) \tag{19}$$

and

$$B(t,x)=\int_0^x \overline{G}(t-y)dG^{(i-1)*}(y),\quad 0<x<t. \tag{20}$$

Taking into consideration formulae (15), (16) and (18) we can state the following theorem giving the representation for the Laplace transform of the queue-size distribution in the system with multiple vacations.

Theorem 2. *For any $\lambda>0$ we have*

$$\int_0^{\infty} e^{-\lambda t}\mathbf{P}\{\xi(t)=m\}dt=\widehat{Q}_{\text{std}}(m,\lambda)+Q(m,\lambda)\int_0^{\infty} e^{-\lambda y}d\Psi(y),\quad m\geq 1 \tag{21}$$

and

$$\int_0^\infty e^{-\lambda t}\mathbf{P}\{\xi(t)=0\}dt = \frac{1-f_1(\lambda)}{\lambda}\int_0^\infty e^{-\lambda y}d\Psi(y), \tag{22}$$

where $\widehat{Q}_{\mathrm{std}}(m,\lambda)$ *and* $Q(m,\lambda)$ *are defined in (10) and (19) respectively.*

5. Representations for distributions of random variables c_i and C_i

The representations for expressions $\widehat{Q}_{\mathrm{std}}(m,\lambda)$ and $\widehat{Q}_n(m,\lambda)$ we can easily obtain using results from [2]. To characterize the distribution of $\xi(t)$ completely we also need formulae for distributions of random variables c_i and C_i (and hence for renewal functions $\Phi(\cdot)$ and $\Psi(\cdot)$). One can find the below formula for $b_0(\cdot)$ for example in [1] or [10]. The formula for $b_1(\cdot)$ was obtained in [10].

The following equalities hold true

$$b_0(\lambda) = \mathbf{E}\{e^{-\lambda c_0}\} = \mathbf{E}\{e^{-\lambda C_0}\} = \mathbf{E}_{\mathrm{std}}\{e^{-\lambda\widehat{\tau}_1}\} = 1 - f_+(\lambda, 0) \tag{23}$$

and

$$b_1(\lambda) = \mathbf{E}\{e^{-\lambda c_1}\} = \sum_{i=1}^\infty\sum_{n=i}^\infty p_n^{i*}\mathbf{E}_n\{e^{-\lambda\widehat{\tau}_1}\}\int_0^\infty e^{-\lambda y}\big[F_1^{i*}(y) - F_1^{(i+1)*}(y)\big]dG(y)$$
$$+\,\mathbf{E}_{\mathrm{std}}\{e^{-\lambda\widehat{\tau}_1}\}\int_0^\infty e^{-\lambda y}G(y)dF_1(y), \tag{24}$$

where $\mathbf{E}_{\mathrm{std}}\{e^{-\lambda\widehat{\tau}_1}\}$ is defined in (23) and moreover

$$\mathbf{E}_n\{e^{-\lambda\widehat{\tau}_1}\} = f_2^n(\lambda) - f_+(\lambda,0)\int_0^\infty e^{-\lambda y}\int_{-0}^y F_1(y-v)dP_+^{(0)}(\lambda,v)dF_2^{n*}(y). \tag{25}$$

The function $f_+(\lambda,0)$ we obtain from the canonical factorization identity

$$1 - f_1(s)p\big(f_2(\lambda-s)\big) = f_+(\lambda,s)f_-(\lambda,s), \quad 0\le \mathrm{Re}(s)\le\lambda, \tag{26}$$

besides $P_+^{(0)}(\lambda,x) = I\{x>0\} + P_+(\lambda,x)$, where $I\{\mathbb{A}\}$ denotes the indicator of event $\mathbb{A}$, and $P_+(\lambda,x)$ is defined by the equation

$$\frac{1}{f_+(\lambda,s)} = 1 + \int_0^\infty e^{-sx}dP_+(\lambda,x). \tag{27}$$

The expressions $\mathbf{E}_{\mathrm{std}}\{e^{-\lambda\widehat{\tau}_1}\}$ and $\mathbf{E}_n\{e^{-\lambda\widehat{\tau}_1}\}$ denote Laplace transforms of d.fs of busy period $\widehat{\tau}_1$ in the "ordinary" system respectively: in the "standard" regime and with n customers present at time $t = 0+$. Finally, we will find the formula for $b_2(\cdot)$. We have

$$b_2(\lambda) = \int_0^\infty e^{-\lambda t} dB_2(t) = \mathbf{E}\{e^{-\lambda C_1}\} = \sum_{j=1}^\infty \int_0^\infty dF_1(y) \int_y^\infty e^{-\lambda z} \times$$

$$\times \sum_{k=1}^\infty \sum_{l=k}^\infty p_l^{k*} \mathbf{E}_l\{e^{-\lambda \widehat{\tau}_1}\} \left[F_1^{k*}(z-y) - F_1^{(k+1)*}(z-y)\right] dG^{j*}(z), \tag{28}$$

where $\mathbf{E}_l\{e^{-\lambda \widehat{\tau}_1}\}$ is defined in (25).

References

[1] Bratiychuk M.S., Kempa W.M., *Application of the superposition of renewal processes to the study of batch arrival queues*, Queueing Systems 2003, 44, 51-67.

[2] Bratiychuk M.S., Kempa W.M., *Explicite formulae for queue lenght of batch arrival systems*, Stochastic Models 2004, 20(4), 457-472.

[3] Kempa W.M., *The virtual waiting time for the batch arrival queueing systems*, Stochastic Analysis and Applications 2004, 22(5), 1235-1255.

[4] Kempa W.M., *The departure process for the queueing systems with batch arrival of customers*, Stochastic Models 2008, 24(2), 246-263.

[5] Baba Y., On the $M^X/G/1$ queue with vacation time, Oper. Res. Lett. 1986, 5, 93-98.

[6] Chatterjee U., Mukherjee S.P., *On some distributions of the $M^X/G/1$ queue with server's vacation and exhaustive service discipline*, Asis-Pac. J. Oper. Res. 1990, 7(1), 82-91.

[7] Choudhury G., *A bath arrival queue with a vacation time under single vacation policy*, Comput. Oper. Res. 2002, 29(14), 1941-1955.

[8] Choudhury G., Kalita S., *Analysis of a bath arrival Poisson queue under single vacation policy*, Calcutta Stat. Assoc. Bull. 2002, 53(209-210), 81-91.

[9] Takagi H., *Queueing Analysis, Volume 1: Vacation and Priority Systems*, Part 1, North-Holland, Amsterdam-London-New York-Tokyo 1983.

[10] Kempa W.M., *Some new results for departure process in $GI^X/G/1$ queueing system with a single vacation time and exhaustive service*, Stochastic Analysis and Applications (submitted in 2008).

[11] Kempa W.M., *$GI/G/1/\infty$ batch arrival queueing system with a single exponential vacation*, Mathematical Methods of Operations Research 2008 (accepted in 2007).