# APPLICATION OF THE LIPSCHITZ EXPONENT AND THE WAVELET TRANSFORM TO FUNCTION DISCONTINUITY ESTIMATION

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**Abstract.** The paper concerns the problem of the application of the Discrete Wavelet Transform and the Lipschitz exponent to an estimation of function differentiability. The influence of number of discrete data (measurements points) and a class of function on the discontinuity indicator is analyzed. The problem is discussed on the example of functions which represent a structural response of a mechanical static system.

## Introduction

The wavelet transform is widely used in many problems like compression and recognizing of images, signal denoising, solving boundary value problem and damage detection.

Application of the wavelet transform to the problem of damage detection is presented in many papers [1-3]. The wavelet transform allows for a multi-resolution analysis of an arbitrary function. Therefore, each level of the transform expresses respective detailed information about the function. The most detailed information usually indicates a position of a signal disturbance. The wavelet transform was originally used in time-frequency domain, but it was quickly found further applications [4-6].

The Lipschitz exponent is a well known tool, which is used to estimate of function differentiability. To find function discontinuity, the exponent is applied together with wavelet transform. For better understanding of our paper, basis of wavelet transform and the Lipschitz exponent are shortly presented below.

The possibility and suitability of function discontinuity estimation using Discrete Wavelet Transform (DWT) is discussed in the paper. A problem of number and density of the data arises because the DWT uses a discrete data representing a function. The problem is analyzed for various classes of function and the results are compared to analytical solutions.

## 1. Discrete wavelet transform and Lipschitz exponent

Wavelet transform was described in many papers [7-9]. The wavelet transform of a function $f(x)$ is defined as a set of coefficients (wavelet coefficients) $d_{j,k}$:

$$d_{j,k} = W\, f(x) = \int_{-\infty}^{+\infty} f(x)\overline{\psi}_{j,k}(x)dx \tag{1}$$

where the term $\overline{\psi}_{j,k}(x)$ denotes the complex conjugate to the wavelet family $\psi_{j,k}$. The family is generated from the mother wavelet $\psi$ according to the expression

$$\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j \cdot x - k\right) \tag{2}$$

The integers $j$ and $k$ are called dilation (scale) and translation (position) parameters, respectively. The scale parameter corresponds to a respective part (level) of function decomposition ($j = 0,1,\ldots, J-1$), where $J$ is a maximum level of the transform. The parameter $k = 0,\ldots,2^{j-1}$ indicates the position of wavelet. A discrete signal $f_J$ (number of discrete data is equal to $2^J$) can be decomposed into the wavelet series

$$f_J(x) = \sum_{j=0}^{J-1}\sum_{k} d_{j,k}\psi_{j,k}(x) \tag{3}$$

A function $f(x)$ is Lipschitz $\alpha \geq 0$ at the point $x = \nu$ if there exists a constant $A > 0$ and a polynomial $p_\nu(x)$ of order $m$ such that

$$f(x) = p_\nu(x) + \varepsilon_\nu(x) \qquad \text{and} \qquad |\varepsilon_\nu(x)| \leq A|x-\nu|^\alpha \tag{4}$$

The term $\varepsilon_\nu(x)$ contains all discontinuities of the function $f(x)$ at the point $x = \nu$.

The Lipschitz exponent (LE) $\alpha$ is a measure of function differentiability at the point $x = \nu$. If the function is not differentiable, then $0 < \alpha < 1$. If the Lipschitz exponent is used to the wavelet transform, then vanishing moments of the mother wavelet play an important role. A wavelet $\psi(x)$ has $n$ vanishing moments if it satisfies

$$\int_{-\infty}^{+\infty} x^k \psi(x)dx \qquad \text{for} \qquad 0 \leq k \leq n \tag{5}$$

If the wavelet function has a sufficient number of vanishing moments such that $n \geq \alpha$, the wavelet transform indicates a singular part of a function because of:

$$Wf(x) = W\varepsilon_\nu(x), \qquad Wp_\nu(x) = 0 \tag{6}$$

The Lipschitz exponent is calculated using the following inequality:

$$\left|W\ f(x)\right| \leq A \cdot 2^{-j(\alpha+1/2)} \tag{7}$$

Hence,

$$\alpha \geq -\frac{\log_2 \left|W\ f(x)\right|}{j} - \frac{1}{2} \tag{8}$$

Application of the Lipschitz exponent was presented in [10, 11].

## 2. Formulation of the problem

Consider a static or dynamic mechanical system subjected to arbitrary actions. Assume that the system contains local information which is hidden in the global response. The response can be treated as a signal in which local disturbances may occur. The main aim is finding these disturbances and estimating the level of discontinuity of the signal. The problem is well placed in the subject area of structural damage identification, since defects generate discontinuities in the response of a system. To solve the problem, wavelet transform connected with the Lipschitz exponent can be used. Unfortunately, in real systems, it is usually not possible to find an exact analytical form of the system response because of considerable limitations to number of measurement points.

Our aim is verification of application possibility of DWT and LE to estimation of function discontinuity in case of the analysis of discrete signal representation. From our point of view a minimum number of measurement points and a type of a wavelet in context of a class of function continuity seems very important. For the simplicity of the presentation, the problem is discussed on the example of a simple mechanical system and its response.

## 3. Examples

In the example, we analyse the structural response of the simply supported beam with span *L*, loaded by distributed force *q* and concentrated force *P* located in the middle of the system (Fig. 1).

Bending stiffness of the beam is equal to *EI*. The functions of shear force *Q*(*x*), moment *M*(*x*), slope *w'*(*x*) and displacement *w*(*x*) of the Bernoulli's beam have simply analytical form:

$$Q(x) = \frac{qL}{2} + \frac{P}{2} - qx - P\left(x - \frac{L}{2}\right)^0 \Phi\left(x - \frac{L}{2}\right) \tag{9}$$

$$M(x)=\frac{qLx}{2}+\frac{Px}{2}-\frac{qx^2}{2}-P\left(x-\frac{L}{2}\right)^1\Phi\left(x-\frac{L}{2}\right) \tag{10}$$

$$EIw'(x)=\frac{qL^3}{24}+\frac{PL^2}{16}-\frac{qLx^2}{4}-\frac{Px^2}{4}+\frac{qx^3}{6}+\frac{P}{2}\left(x-\frac{L}{2}\right)^2\Phi\left(x-\frac{L}{2}\right) \tag{11}$$

$$EIw(x)=\frac{qL^3x}{24}+\frac{PL^2x}{16}-\frac{qLx^3}{12}-\frac{Px^3}{12}+\frac{qx^4}{24}+\frac{P}{6}\left(x-\frac{L}{2}\right)^3\Phi\left(x-\frac{L}{2}\right) \tag{12}$$

where $\Phi(x)$ is Heaviside's function. The functions $M(x)$, $w'(x)$ and $w(x)$ are the class $C^0$, $C^1$ and $C^2$, respectively (Fig. 2). In our example we assume the following parameters: $L$ = 5.12 m, q = 10 kN/m, P = 10 kN and EI = 5000 kNm$^2$.

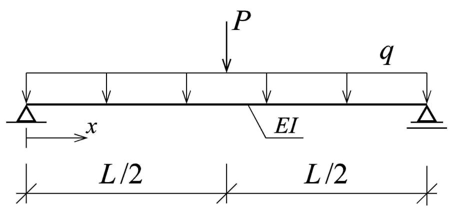



Fig. 1. The model of the simply supported beam structure

The discontinuity of the shear force function is clearly visible. Application of DWT (Daublet 8 mother wavelet, 1024 points) to $M(x)$ reveal local disturbances of the function in the middle of the structure. The phenomena can be observed in the transform details $D_1$, $D_2$ and others (Fig. 3).

Applied the Lipschitz exponent, we can find the parameter α, which in this case is equal to 1.0226 (Fig. 4). It means that the function $M(x)$ is continuous and its derivative is not continuous. Please note, that used in the example wavelet daublet 8 has four vanishing moments and therefore it is able to verify continuity of three derivatives of the function.

Now we can focus on the problem of number of measurement points. Above described approach was used in analysis of displacement and slope functions for N = 1024, 256 and 64 points. The wavelets from the same family but with various number of vanishing moments were used, namely daublet 2, 4, 6, 8 and daublet 10. They have from 1 to 5 vanishing moments, respectively. The values of received Lipschitz exponents α are presented in the Table 1.

The results proof that Lipschitz exponent together with DWT can be successfully applied in the estimation of function differentiability. Type of wavelet should be suitable to detect local discontinuities. For example, in case of the function $w(x)$, the admissible wavelet is daublet 6, but just daublet 8 shows that there are only two and no more continuous derivatives. Received values for inadmissible wavelets were shaded in the table. Interesting is also that if we use very poor wavelet like daublet 2, it is impossible to answer for the question about continuity level of $w(x)$. Please also note, that the size of discontinuity (in our situation value of the force $P$) has almost no importance on the value of $\alpha$.

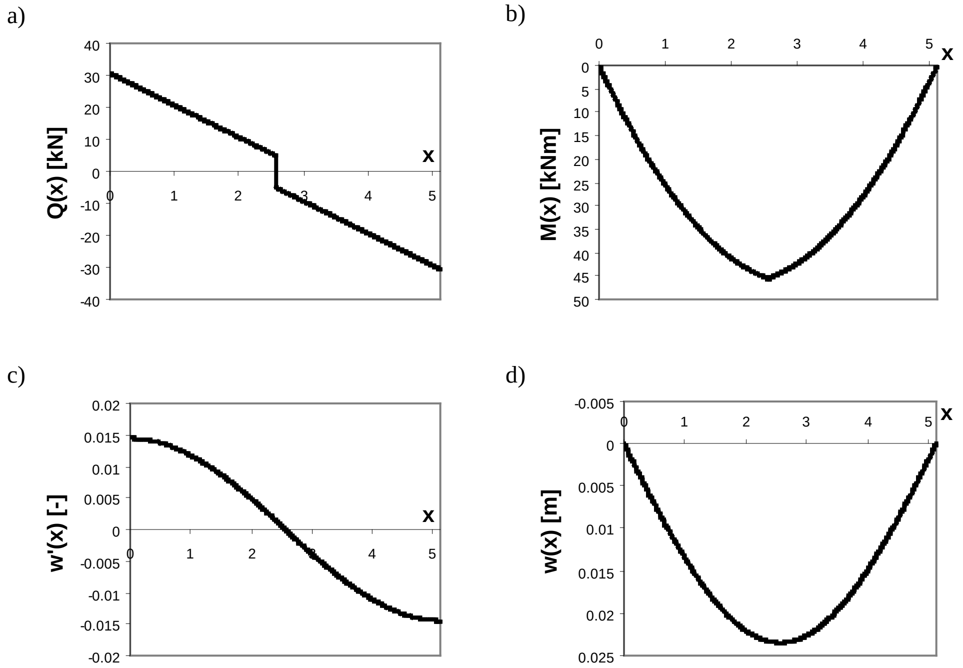



Fig. 2. The structural response of the beam: a) shear force, b) bending moment, c) slope, d) displacement function

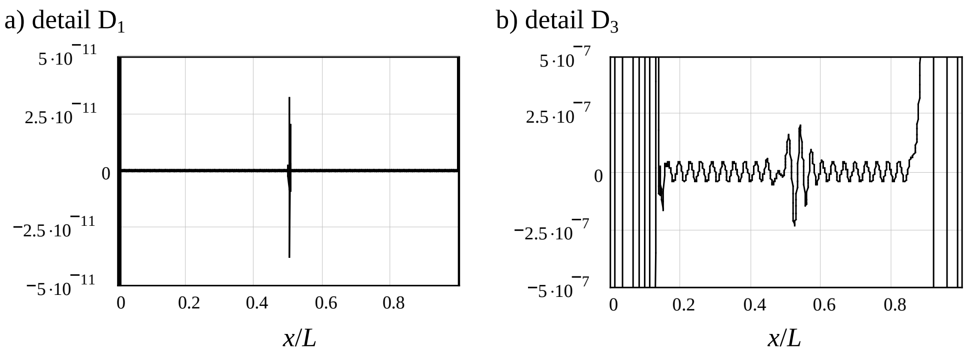
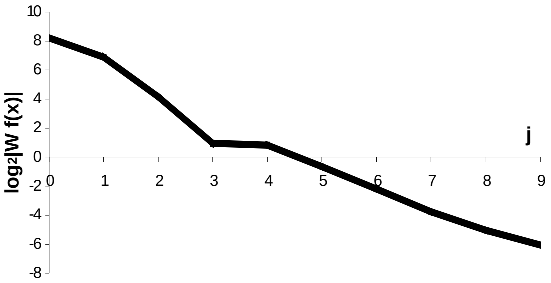



Fig. 3. Details $D_1$ and $D_3$ of DWT (Daublet 8, N = 1024) of $M(x)$




$$\alpha = -\frac{\log_2|Wf(j=9)| - \log_2|Wf(j=5)|}{9-5}$$

$$\alpha = 1.0226$$

Fig. 4. The diagram of $\log_2|Wf(x = \nu)|$ as a function of wavelet transform level $j$

Table 1

**The Lipschitz exponent $\alpha$ of the functions $w(x)$ and $w'(x)$**

| Function | Number of points N | Wavelet function | | | | |
|---|---|---|---|---|---|---|
| | | Daublet 2 | Daublet 4 | Daublet 6 | Daublet 8 | Daublet 10 |
| $w(x)$ | 1024 | - | 1.998 | 3.1201 | 3.0755 | 3.0001 |
| | 256 | - | 1.9697 | 3.2446 | 3.1303 | 3.0968 |
| | | | | | | |
| $w'(x)$ | 1024 | 0.9980 | 1.9037 | 1.9934 | 2.0681 | 1.9635 |
| | 256 | 0.9615 | 2.2859 | 2.0849 | 2.0681 | 1.9635 |

The value of the Lipschitz exponent depends on approximation used in calculation of the exponent. The relation between number of discrete data and the exponent is rather clear. If there are more points, function continuity is lower because when points are very closed to each other, discontinuity is "visible" as more sharp. Trials of signal analysis basing on 64 points ended in failure. The minimum number of points is equal to 128, however in practise, a measurement noise may highly influence on results.

## Concluding remarks

The presented analyses demonstrate that the Lipschitz exponent and the Discrete Wavelet Transform can be successfully used in the estimation of function differentiability. The type of applied wavelet functions (in fact number of vanishing moments) should depend on the type of signal discontinuities. The number of discrete data influences on the results, but the differences are relatively small. It is important that the proper analysis requires certain number of measurement points. The effectiveness and simplicity of the method make it a powerful tool in signal discontinuity analysis.

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