# PARAMETRIZATIONS OF INTEGRALS

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**Abstract.** In the present paper I give parametric formulas of integrals of meromorphic forms in the case of $C^2$.

## 1. Parametrizations of integrals at finity

Integrals of meromorphic forms occur in the definition of residue. Let us remind then the definition of residue of holomorphic mapping at a point. To retain symmetry with second part of this paper we will limit to the case of $\boldsymbol{C}^2$. Let $f = (f_1, f_2)$ be holomorphic mapping in the neighbourhood of point $\alpha = (\alpha_1, \alpha_2) \in \boldsymbol{C}^2$ with zero isolated at this point; and $g$ holomorphic function in the neighbourhood of point $\alpha$. As residue of pair $g, f$ at point $a$ we define an integral of the form (s.[1, 2])

$$\operatorname{Res}_a g \big/ f = \frac{1}{(2\pi i)^2} \int_{\Gamma_a} \frac{g(z)\, dz_1 \wedge dz_2}{f_1(z) \cdot f_2(z)}$$

where $\Gamma_a = \{z : |f_1(z)| = \varepsilon, |f_2(z)| = \varepsilon\}$ is sufficiently small real two-cycle in the neighbourhood of point $\alpha$ with positive orientation given by nowhere not disappearing on $\Gamma_\alpha$ form $d(\arg f_1) \wedge d(\arg f_2)$.

Calculating of residue we might then reduce to calculus of residues of meromorphic functions of one variable. However if the germ of function $f_1$ in point $\alpha$ has reduced decomposition then

$$f_1(z) = f_{11}(z) \ldots f_{1m}(z)$$

for $z$ from some neighbourhood of point $\alpha$. Then (s. [3])

$$\operatorname{Res}_a g \big/ f = \sum_{1 \le j \le m} \operatorname{res}_0 \frac{g \circ \Phi_j}{\operatorname{Jac} f \circ \Phi_j} \frac{(f_2 \circ \Phi_j)'}{f_2 \circ \Phi_j}$$

where $\Phi_j$ is parametrization of the set of zeros of function $f_{1j}$ defined in the neighbourhood of point 0 at $\boldsymbol{C}$, $\Phi_j(0) = \alpha$, $j = 1,\ldots, m$, a Jac$f$ denote a Jacobian of the mapping $f$. Thus the integral of meromorphic two-form is reduced to the integrals of meromorphic functions

$$\int_{\Gamma_a} \frac{g(z)dz_1 \wedge dz_2}{f_1(z)\cdot f_2(z)} = 2\pi i \sum_{1\le j\le m} \int_{C_j} \frac{g(\Phi_j(t))}{\operatorname{Jac} f(\Phi_j(t))} \frac{f_2(\Phi_j(t))'}{f_2(\Phi_j(t))} dt$$

where $C_j$ are sufficiently small positively oriented circles with the center in point 0 at $\boldsymbol{C}$. Similarly, if the germ of function $f_2$ at point $\alpha$ has reduced decomposition, then

$$f_2(z) = f_{21}(z)\ldots f_{2n}(z)$$

for $z$ from some neighbourhood of point $\alpha$, thus

$$\int_{\Gamma_a} \frac{g(z)dz_1 \wedge dz_2}{f_1(z)\cdot f_2(z)} = 2\pi i \sum_{1\le k\le n} \int_{C_k} \frac{g(\Psi_k(t))}{\operatorname{Jac} f(\Psi_k(t))} \frac{f_1(\Psi_k(t))'}{f_1(\Psi_k(t))} dt$$

where $\Psi_k$ is parametrization of the set of zeros of function $f_{2k}$ defined in the neighbourhood of point 0 at $\boldsymbol{C}$, $\Psi_k(0) = \alpha$, $k = 1,\ldots, n$.
Applying above parametric formulas we obtain the given relation between integrals of following two-forms (s. [4])

$$(*)\quad \int_{\Gamma_a} \frac{g(z)dz_1 \wedge dz_2}{f_1(z)\cdot (z_1-a_1)^\sigma f_2(z)} - \int_{\Gamma_a} \frac{g(z)dz_1 \wedge dz_2}{(z_1-a_1)^\sigma f_1(z)\cdot f_2(z)} = \int_{\Gamma_a} \frac{g(z)dz_1 \wedge dz_2}{f_1 f_2(z)\cdot (z_1-a_1)^\sigma}$$

$$\text{for } \sigma \ge 0$$

## 2. Parametrizations of integrals at infinity

Integrals of rational forms occurs in definition of residue at infinity. At the beginning let us assume the following definitions. For polynomial $h$ of two variables we define polynomial

$$\tilde{h}(X_1, X_2) = X_1^{\deg h} h\left(\frac{1}{X_1}, \frac{X_2}{X_1}\right)$$

and for point $p = (0:1:y) \in \boldsymbol{P}^2$ its affine image $\tilde{p} = (0, y) \in \boldsymbol{C}^2$.

Let $f = (f_1, f_2)$ be polynomial defined on $\boldsymbol{C}^2$ of components relatively prime and different then constants while $g$ be arbitrary polynomial of two variables. Let us denote $\sigma = \deg f_1 + \deg f_2 - \deg g - 3$. The residue of pair $g, f$ at infinity we define by the formula (s. [4, 5])

$$\operatorname{Res}_\infty g\big/f = -\sum_{c\in(C_1\cap C_2)\cap l_\infty} \operatorname{Res}_{\tilde{c}} \tilde{g}X_1^{\sigma}\big/\left(\tilde{f}_1, \tilde{f}_2\right) \quad \text{for } \sigma \geq 0$$

and

$$\operatorname{Res}_\infty g\big/f = -\sum_{a\in C_1\cap l_\infty} \operatorname{Res}_{\tilde{a}} \tilde{g}\big/\left(\tilde{f}_1, X_1^{-\sigma}\tilde{f}_2\right) = -\sum_{b\in C_2\cap l_\infty} \operatorname{Res}_{\tilde{b}} \tilde{g}\big/\left(X_1^{-\sigma}\tilde{f}_1, \tilde{f}_2\right)$$

$$\text{for } \sigma < 0$$

where $l_\infty$ represents the line at infinity over $\boldsymbol{C}^2$, while $C_1$ i $C_2$ are the closers at $\boldsymbol{P}^2$ of curves $f_1 = 0$ and $f_2 = 0$, respectively. In the second part of definition we additionally assume that $(0:0:1) \notin C_1 \cap l_\infty$ and $(0:0:1) \notin C_2 \cap l_\infty$, what in fact just simplifies the notation (s. [4]). The integrals of forms occurring in expression of residue at infinity we may now parametrize. Let $\sigma \leq 0$ and let $c \in (C_1 \cap C_2) \cap l_\infty$. If the germ of function $\tilde{f}_1$ at point $\tilde{c}$ has a reduced decomposition, then

$$\tilde{f}_1(x) = \tilde{f}_{11}(x) \ldots \tilde{f}_{1p}(x)$$

for $x$ from some neighbourhood of point $\tilde{c}$. Then

$$\int_{\Gamma_{\tilde{c}}} \frac{\tilde{g}(x)x_1^{\sigma}dx_1 \wedge dx_2}{\tilde{f}_1(x)\cdot\tilde{f}_2(x)} = 2\pi i \sum_{1\leq j\leq p} \int_{C_j} \frac{\tilde{g}\left(\tilde{\Phi}_j(t)\right)t^{\sigma\mu_j}}{\operatorname{Jac}\tilde{f}\left(\tilde{\Phi}_j(t)\right)} \frac{\tilde{f}_2\left(\tilde{\Phi}_j(t)\right)'}{\tilde{f}_2\left(\tilde{\Phi}_j(t)\right)} dt$$

where $\tilde{\Phi}_j(t) = \left(t^{\mu_j}, \varphi_j(t)\right)$ is parametrization of the set of zeros of function $\tilde{f}_{1j}$ in the neighbourhood of point 0 at $\boldsymbol{C}$, $\boldsymbol{C}, \tilde{\Phi}_j(0) = \tilde{c}$, $j = 1, \ldots, p$, where $\tilde{f} = \left(\tilde{f}_1, \tilde{f}_2\right)$. Similarly, if the germ of function $\tilde{f}_2$ at point $\tilde{c}$ has reduced decomposition, then

$$\tilde{f}_2(x) = \tilde{f}_{21}(x) \ldots \tilde{f}_{2q}(x)$$

for $x$ from some neighbourhood of point $\tilde{c}$. Then

$$\int_{\Gamma_{\tilde{c}}} \frac{\tilde{g}(x)x_1^{\sigma}dx_1 \wedge dx_2}{\tilde{f}_1(x)\cdot\tilde{f}_2(x)} = 2\pi i \sum_{1\leq k\leq q} \int_{C_k} \frac{\tilde{g}\left(\tilde{\Psi}_k(t)\right)t^{\sigma\nu_k}}{\operatorname{Jac}\tilde{f}\left(\tilde{\Psi}_k(t)\right)} \frac{\tilde{f}_1\left(\tilde{\Psi}_k(t)\right)'}{\tilde{f}_1\left(\tilde{\Psi}_k(t)\right)} dt$$

where $\widetilde{\Psi}_k(t) = \left(t^{\nu_k}, \widetilde{\psi}_k(t)\right)$ is parametrization of the set of zeros of function $\widetilde{f}_{2k}$ defined in the neighbourhood of point 0 at $\boldsymbol{C}$, $\widetilde{\Psi}_k(0) = \widetilde{c}$, $k = 1,...,q$. The residue at point $c \in (C_1 \cap C_2) \cap l_\infty$ at infinity we may, in this case, define as

$$\operatorname{Res}_c g/f = -\frac{1}{(2\pi i)^2} \int_{\Gamma_{\widetilde{c}}} \frac{\widetilde{g}(x) x_1^\sigma dx_1 \wedge dx_2}{\widetilde{f}_1(x) \cdot \widetilde{f}_2(x)}$$

Then

$$\operatorname{Res}_\infty g/f = \sum_{c \in (C_1 \cap C_2) \cap l_\infty} \operatorname{Res}_c g/f \quad \text{for } \sigma \geq 0$$

Let $\sigma < 0$ and let $a \in C_1 \cap l_\infty$. If the germ of function $\widetilde{f}_1$ at point has the reduced decomposition, then

$$\widetilde{f}_1(x) = \widetilde{f}_{11}(x) \ldots \widetilde{f}_{1r}(x)$$

for $x$ from some neighbourhood of point $\widetilde{a}$. Then

$$\int_{\Gamma_{\widetilde{a}}} \frac{\widetilde{g}(x) dx_1 \wedge dx_2}{\widetilde{f}_1(x) \cdot x_1^{-\sigma} \widetilde{f}_2(x)} = 2\pi i \sum_{1 \leq j \leq r} \int_{C_j} \frac{\widetilde{g}\left(\widetilde{\Phi}_j(t)\right)}{\operatorname{Jac} \widetilde{f}_I\left(\widetilde{\Phi}_j(t)\right)} \left( \frac{\left(\widetilde{f}_2\left(\widetilde{\Phi}_j(t)\right)\right)'}{\widetilde{f}_2\left(\widetilde{\Phi}_j(t)\right)} - \frac{\sigma \mu_j}{t} \right) dt$$

where $\widetilde{\Phi}_j(t) = \left(t^{\mu_j}, \widetilde{\varphi}_j(t)\right)$ is parametrization of the set of zeros of function $\widetilde{f}_{1j}$ defined in the neighbourhood of point 0 at $\boldsymbol{C}$, $\widetilde{\Phi}_j(0) = \widetilde{\alpha}$, and $\widetilde{f}_I = \left(\widetilde{f}_1, X_1^{-\sigma} \widetilde{f}_2\right)$. Then the residue at point $a \in C_1 \cap l_\infty$ at infinity we may define as

$$\operatorname{Res}_a^I g/f = -\frac{1}{(2\pi i)^2} \int_{\Gamma_{\widetilde{a}}} \frac{\widetilde{g}(x) dx_1 \wedge dx_2}{\widetilde{f}_1(x) \cdot x_1^{-\sigma} \widetilde{f}_2(x)}$$

Then

$$\operatorname{Res}_\infty g/f = \sum_{a \in C_1 \cap l_\infty} \operatorname{Res}_a^I g/f \quad \text{for } \sigma < 0$$

Similarly, let $\sigma < 0$ and let $b \in C_2 \cap l_\infty$. If the germ of function $\widetilde{f}_2$ at point $\widetilde{b}$ has reduced decomposition, then

$$\widetilde{f}_2(x) = \widetilde{f}_{21}(x) \ldots \widetilde{f}_{2s}(x)$$

for $x$ from some neighbourhood of point $\widetilde{b}$. Then

$$\int_{\Gamma_{\widetilde{b}}} \frac{\widetilde{g}(x)dx_1 \wedge dx_2}{x_1^{-\sigma}\widetilde{f}_1(x)\cdot\widetilde{f}_2(x)} = 2\pi i \sum_{1\le k\le s} \int_{C_k} \frac{\widetilde{g}\left(\widetilde{\Psi}_k(t)\right)}{\operatorname{Jac}\widetilde{f}_{II}\left(\widetilde{\Psi}_k(t)\right)} \left( \frac{\widetilde{f}_1\left(\widetilde{\Psi}_k(t)\right)'}{\widetilde{f}_1\left(\widetilde{\Psi}_k(t)\right)} - \frac{\sigma\nu_k}{t} \right) dt$$

where $\widetilde{\Psi}_k(t) = \left(t^{\nu_k}, \widetilde{\psi}_k(t)\right)$ is parametrization of the set of zeros of function $\widetilde{f}_{2k}$ defined in the neighbourhood of point 0 in $\boldsymbol{C}$, $\widetilde{\Psi}_k(0) = \widetilde{b}$, $k = 1,\dots,s$, and $\widetilde{f}_{II} = \left(X_1^{-\sigma}\widetilde{f}_1, \widetilde{f}_2\right)$. Then the residue at point $b \in C_2 \cap l_\infty$ at infinity we may define as

$$\operatorname{Res}_b^{II} g/f = -\frac{1}{(2\pi i)^2} \int_{\Gamma_{\widetilde{b}}} \frac{\widetilde{g}(x)dx_1 \wedge dx_2}{x_1^{-\sigma}\widetilde{f}_1(x)\cdot\widetilde{f}_2(x)}$$

Then

$$\operatorname{Res}_\infty g/f = \sum_{b\in C_2\cap l_\infty} \operatorname{Res}_b^{II} g/f \quad \text{for } \sigma < 0$$

Let us now observe that if $a = b$ and $\sigma < 0$, then the residues I and II are connected by equation

(**) $$\operatorname{Res}_a^{I} g/f - \operatorname{Res}_a^{II} g/f = \frac{1}{(2\pi i)^2} \int_{\Gamma_{\widetilde{a}}} \frac{\widetilde{g}(x)dx_1 \wedge dx_2}{\widetilde{f}_1\widetilde{f}_2(x)\cdot x_1^{-\sigma}}$$

## References


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