Please cite this article as:
Grzegorz Biernat, The residue at infinity and Bezout's theorem, Scientific Research of the Institute of Mathematics and Computer Science, 2002, Volume 1, Issue 1, pages 25-27.
The website: http://www.amcm.pcz.pl/

THE RESIDUE AT INFINITY AND BEZOUT'S THEOREM

Grzegorz Biernat

Institute of Mathematics and Computer Science, Technical University of Częstochowa

Abstract. In this paper we give an alternative proof, based on properties of the residue at infinity, of Bezout's theorem in $\boldsymbol{C}^2$.

1. The residue at infinity

Let $l_\infty = V(T_0)$ denote a line at infinity in the projective complex space $\boldsymbol{P}^2$ (with homogeneous coordinates $T_0 : T_1 : T_2$). Further it will be called infinity. If $a \in l_\infty$ then by $\tilde{a} \in \boldsymbol{C}^2$ we denote the canonical image of the point a in affine part $\boldsymbol{P}^2 - V(T_1) \cong \boldsymbol{C}^2$. For a polynomial h of two variables, $\tilde{h}$ signifies a suitable dehomogenization of the homogenization of the polynomial h. So, we have $\tilde{h}(X_1, X_2) = X_1^{\deg h} h(1/X_1, X_2/X_1)$.

Let f_1, f_2 and g be polynomials of two variables and let C_1, C_2 be the closures of the curves $V(f_1) = \{z \in \boldsymbol{C}^2 : f_1(z) = 0\}, V(f_2) = \{z \in \boldsymbol{C}^2 : f_2(z) = 0\}$ in the space $\boldsymbol{P}^2$. Assume further that polynomials f_1 and f_2 are different from constants and have not common factors of positive degrees. Put $s = \deg f_1 + \deg f_2 - \deg g - 3$ and let $f = (f_1, f_2)$. We define (see [1-3])

$$\operatorname{Res}_\infty g/f = \operatorname{Res}_\infty g/(f_1, f_2) = \begin{cases} -\sum\limits_{c \in (C_1 \cap C_2) \cap l_\infty} \operatorname{Res}_{\tilde{c}}\, \tilde{g} X_1^s / (\tilde{f}_1, \tilde{f}_2) & \text{if } s \geq 0 \\ -\sum\limits_{a \in C_1 \cap l_\infty} \operatorname{Res}_{\tilde{a}}\, \tilde{g} / (\tilde{f}_1, X_1^{-s} \tilde{f}_2) & \text{if } s < 0 \end{cases}$$

$$\text{and } (0:0:1) \notin C_1 \cap l_\infty$$

This number will be called the residue at infinity of the pair g, f.

2. Application

Further we denote by $J_f = \operatorname{Jac}(f_1, f_2)$ (respectively, $J_{\tilde{f}} = \operatorname{Jac}(\tilde{f}_1, \tilde{f}_2)$) the jacobian of the mapping $f = (f_1, f_2)$ (respectively, $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$). The well-know Euler identity for homogeneous forms easily implies the following formula

$$X_1^{\sigma}\widetilde{J}_f + X_1 J_{\widetilde{f}} = \det\begin{pmatrix} n_1\widetilde{f}_1 & \dfrac{\partial\widetilde{f}_1}{\partial X_2} \\ n_2\widetilde{f}_2 & \dfrac{\partial\widetilde{f}_2}{\partial X_2}\end{pmatrix} \qquad (*)$$

where $n_1 = deg\, f_1$, $n_2 = \deg f_2$ and $\sigma = n_1 + n_2 - 2 - deg\, J_f \geq 0$.

Lemma. *If the polynomials f_1 and f_2 have not common factor of positive degree, then*

$$\operatorname{Res}_\infty J_f \big/ f = \sum_{c\in(C_1\cap C_2)\cap l_\infty} \operatorname{mult}_{\widetilde{c}} \widetilde{f} - n_1 n_2$$

We adopt the first component of the right-hand side equal to zero when $(C_1 \cap C_2)\cap l_\infty = \varnothing$.

Proof. The transformation principle, formula (*) and the elementary properties of the residues (see [3]) allow us to write

$$\operatorname{Res}_\infty J_f / f = -\sum_{a\in C_1\cap l_\infty} \operatorname{Res}_{\widetilde{a}} X_1^{\sigma}\widetilde{J}_f \big/ \left(\widetilde{f}_1, X_1\widetilde{f}_2\right) =$$

$$-\sum_{a\in C_1\cap l_\infty} \operatorname{Res}_{\widetilde{a}} \left(\det\begin{pmatrix} n_1\widetilde{f}_1 & \dfrac{\partial\widetilde{f}_1}{\partial X_2} \\ n_2\widetilde{f}_2 & \dfrac{\partial\widetilde{f}_2}{\partial X_2}\end{pmatrix} - X_1 J_{\widetilde{f}}\right) \Bigg/ \left(\widetilde{f}_1, X_1\widetilde{f}_2\right) =$$

$$n_2 \sum_{a\in C_1\cap l_\infty} \operatorname{Res}_{\widetilde{a}} \widetilde{f}_2 \frac{\partial\widetilde{f}_1}{\partial X_2} \Bigg/ \left(\widetilde{f}_1, X_1\widetilde{f}_2\right) + \sum_{a\in C_1\cap l_\infty} \operatorname{Res}_{\widetilde{a}} X_1 J_{\widetilde{f}} \big/ \left(\widetilde{f}_1, X_1\widetilde{f}_2\right) =$$

$$n_2 \sum_{a\in C_1\cap l_\infty} \operatorname{Res}_{\widetilde{a}} \frac{\partial\widetilde{f}_1}{\partial X_2} \Bigg/ \left(\widetilde{f}_1, X_1\right) + \sum_{c\in(C_1\cap C_2)\cap l_\infty} \operatorname{Res}_{\widetilde{c}} J_{\widetilde{f}} \big/ \left(\widetilde{f}_1, \widetilde{f}_2\right) =$$

$$\sum_{c\in(C_1\cap C_2)\cap l_\infty} \operatorname{mult}_{\widetilde{c}} \widetilde{f} - n_2 \sum_{a\in C_1\cap l_\infty} \operatorname{Res}_{\widetilde{a}} \operatorname{Jac}\left(X_1, \widetilde{f}_1\right) \big/ \left(X_1, \widetilde{f}_1\right) =$$

$$\sum_{c\in(C_1\cap C_2)\cap l_\infty} \operatorname{mult}_{\widetilde{c}} \widetilde{f} - n_2 \sum_{a\in C_1\cap l_\infty} \operatorname{mult}_{\widetilde{a}} \left(X_1, \widetilde{f}_1\right) = \sum_{c\in(C_1\cap C_2)\cap l_\infty} \operatorname{mult}_{\widetilde{c}} \widetilde{f} - n_2 n_1$$

This ends the proof.

Example. Let $f_1(Y_1, Y_2) = Y_1^2 - Y_2^2$, $f_2(Y_1, Y_2) = Y_1^2 - Y_2^2 - 1$. Then $J_f = 0$ and $(C_1 \cap C_2) \cap l_\infty = \{(0:1:1), (0:1:-1)\}$. We have $\widetilde{f}_1(X_1, X_2) = 1 - X_2^2$, $\widetilde{f}_2(X_1, X_2) = = 1 - X_1^2 - X_2^2$ and $\widetilde{J}_f = 0$. Thus $mult_{(0,1)}(\widetilde{f}_1, \widetilde{f}_2) = 2$, $\mathrm{mult}_{(0,-1)}(\widetilde{f}_1, \widetilde{f}_2) = 2$ and

$$0 = \mathrm{Res}_\infty J_f / f = 2 + 2 - 2 \cdot 2$$

From this lemma and theorem of residues (see [1, 3]) we immediately obtain:

Corollary (Bezout's theorem). *If the polynomials f_1 and f_2 have not common factor of positive degree, then*

$$\sum_{z \in V(f_1) \cap V(f_2)} \mathrm{mult}_z(f_1, f_2) + \sum_{c \in (C_1 \cap C_2) \cap l_\infty} \mathrm{mult}_{\widetilde{c}}(\widetilde{f}_1, \widetilde{f}_2) = \deg f_1 \cdot \deg f_2$$

References

[1] Biernat G., Théoréme des résidus dans $\boldsymbol{C}^2$. Prace Naukowe IMiI, Częstochowa 2002, 19-24.

[2] Biernat G., On the Jacobi-Kronecker formula for a polynomial mapping having zeros at infinity, Bull. Soc. Sci. Lettres Łódź 1992, XLII(29), 103-111.

[3] Griffiths P., Harris J., Principles of Algebraic Geometry, New York 1978.