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MATHEMATICAL MODEL OF WARMING UP A COLD HAND BY CONTACT WITH A HOT PIPE

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Abstract. In this article, the problem of heating a cold hand squeezing a pipe through which hot water flows is considered. The mathematical model concerns a one-dimensional axisymmetric problem in cylindrical coordinates. The aim of the research is to estimate the probability of different degrees of burns depending on the temperature of the flowing hot water, the pipe material, and the heating time. This probability is estimated using the Arrhenius integral. The considered problem is solved numerically using the finite volume method. Numerical examples of simulations for various pipe materials are presented.

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Keywords: *heat conduction, tissue burns, Arrhenius integral*

1. Introduction

Among all the physical phenomena described by scientists over the years, one of the rather frequent ones is the issue of heat conduction. The basis for further, more individual and detailed studies can be found in books [1-3]. Although in recent years there seems to be a particular interest in modeling the phenomenon of heat conduction (and not only) using fractional differential equations [4-6], the basis is still describing these problems using partial differential equations. Methods for solving these equations can be found, for example, in [7, 8]. One of the issues related to heat transfer modeling is the problem of heating biological tissue [9-11]. This problem is most often described by the Pennes equation, which is based on the Fourier's heat equation and includes additional internal heat sources resulting from blood perfusion and metabolism [12, 13].

One of the problems we deal with in winter is trying to warm up our hands when they are freezing cold. The first thing that comes to mind is to warm up the hands by putting them in hot water. However, this is dangerous because it can cause thermal shock and even burst blood vessels. Therefore, in the event of cooling, gradual heating is important. In this article, we consider the problem of heating a cold hand by squeezing a pipe in which hot water flows. We simplify the considerations to a geometrical model, in which we analyse a one-dimensional axisymmetric problem in cylindrical coordinates. The aim of our research is to estimate the probability of a given degree of tissue burn depending on the temperature of the hot water flowing inside the pipe, the type of pipe material, and the time of heating the hand. The degree of burn is examined using the Arrhenius integral. The temperature distribution is determined numerically using the finite volume method. The numerical analysis was performed for two types of pipes – for a steel pipe and a PP-R polypropylene pipe.

The paper is structured as follows: Section 2 presents a mathematical model of the problem – the subject of considerations and governing equations describing the problem under consideration. In particular, a method for determining the degree of tissue burns is given. Section 3 presents numerical examples of determining the temperature distribution for two types of pipe materials. The work ends with Section 4, which presents basic conclusions and possible further research.

2. Mathematical model of the problem

The subject of the considerations is the heating of a hand that squeezes a pipe through which hot water flows. The squeezed hand can be simplified in the geometrical model to a ring, which is shown in Figure 1, where region Ω_0 (i.e. $r \in [0, r_0)$) represents the flowing water, Ω_1 (i.e. $r \in (r_0, r_1)$) represents the pipe, and Ω_2 (i.e. $r \in (r_1, r_2)$) represents the squeezing hand, and Γ_{0-1} (i.e. $r = r_0$), Γ_{1-2} (i.e. $r = r_1$), Γ_b (i.e. $r = r_2$) mean the boundaries of appropriate regions.

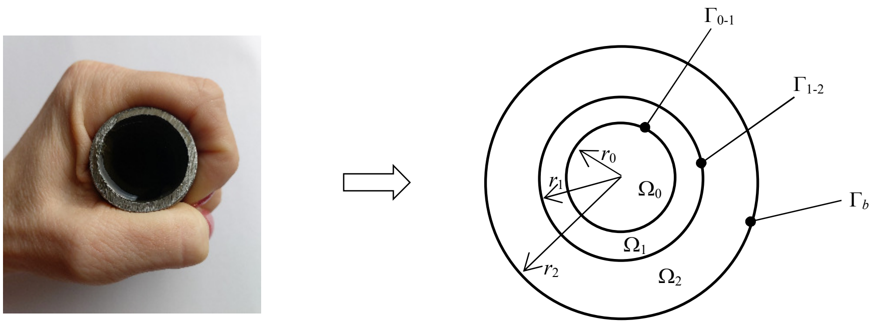

Fig. 1. Sketch of a hand squeezing a pipe

The distribution of the temperature T_0 in region Ω_0 is given by the following relation

$$T_0(r,t) = T_{water} = \text{const} \tag{1}$$

where T_{water} is the temperature of the hot water inside the pipe, and t is time.

The heat conduction equation in the region Ω_1 has the following form

$$c_1\rho_1\frac{\partial T_1(r,t)}{\partial t}=\lambda_1\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1(r,t)}{\partial r}\right) \tag{2}$$

where T_1 is the temperature field of the pipe, c_1 is the specific heat capacity, ρ_1 is the density of the pipe material, and λ_1 is the thermal conductivity of the material.

Whereas, the heat conduction in the region Ω_2 is given by the following Pennes bioheat equation

$$c_2\rho_2\frac{\partial T_2(r,t)}{\partial t}=\lambda_2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2(r,t)}{\partial r}\right)+Q_{met}+Q_{perf}(r,t) \tag{3}$$

where T_2 is the temperature field of the squeezed hand, Q_{met} is the metabolic heat source, and Q_{perf} is the perfusion heat source, which is given by the relationship

$$Q_{perf}(r,t)=c_{bl}\rho_{bl}G_{bl}\left(T_{bl}-T_2(r,t)\right) \tag{4}$$

where T_{bl} is the temperature of the blood, c_{bl} is the specific heat capacity of the blood, ρ_{bl} is the density of the blood and G_{bl} is the blood perfusion rate.

The boundary conditions are as follows:

- on the boundary Γ_{0-1} (at the interface between the water and the pipe):

$$T_1(r,t)\big|_{r=r_0}=T_{water}, \tag{5}$$

- on the boundary Γ_{1-2} (at the interface of the pipe and the hand):

$$T_1(r,t)\big|_{r=r_1}=T_2(r,t)\big|_{r=r_1}, \tag{6}$$

$$-\lambda_1\frac{\partial T_1(r,t)}{\partial r}\bigg|_{r=r_1}=-\lambda_2\frac{\partial T_2(r,t)}{\partial r}\bigg|_{r=r_1}, \tag{7}$$

- on the boundary Γ_b (at the outer surface of the hand):

$$-\lambda_2\frac{\partial T_2(r,t)}{\partial r}\bigg|_{r=r_2}=\alpha\left(T_2(r,t)\big|_{r=r_2}-T_{amb}\right) \tag{8}$$

where α is the outer heat transfer coefficient and T_{amb} is the ambient temperature (e.g. room temperature).

The initial conditions have the forms:

$$T_1(r,0) = T_{init,1}(r), \quad r \in (r_0, r_1) \tag{9}$$

$$T_2(r,0) = T_{init,2}(r), \quad r \in (r_1, r_2) \tag{10}$$

where we assume that the initial temperatures $T_{init,1}$ and $T_{init,2}$ are solutions of the appropriate stationary heat conduction equations.

2.1. Determination of the initial temperature $T_{init,1}(r)$

We begin our considerations with the case when hot water flows through a pipe that is not touched. A representative photo of this case along with a simplified geometric model are presented in Figure 2.

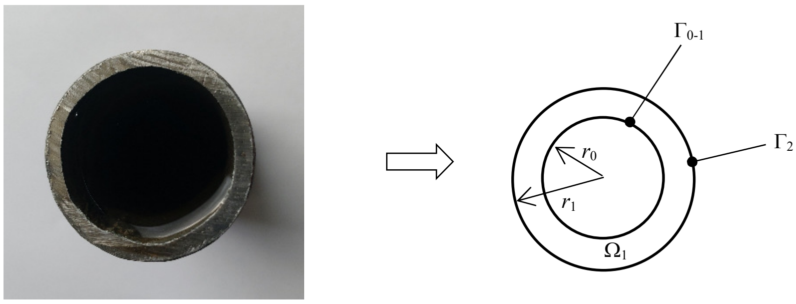

Fig. 2. Sketch of a pipe with water

The heat conduction equation in the region Ω_1 has the following form:

$$\lambda_1 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T_{init,1}(r)}{\partial r} \right) = 0. \tag{11}$$

The equation (11) is complemented by the following boundary conditions

- on the boundary Γ_{0-1}, i.e. for $r = r_0$:

$$T_{init,1}(r)\big|_{r=r_0} = T_{water}, \tag{12}$$

- on the boundary Γ_2, i.e. for $r = r_1$:

$$-\lambda_1 \frac{\partial T_{init,1}(r)}{\partial r}\bigg|_{r=r_1} = \alpha\left(T_{init,1}(r)\big|_{r=r_1} - T_{amb} \right). \tag{13}$$

2.2. Determination of the initial temperature $T_{init,2}\left(r\right)$

Next, we consider the case of the free hand held in a cool environment, and this situation is shown in Figure 3, wherein $L = r_2 - r_1$.

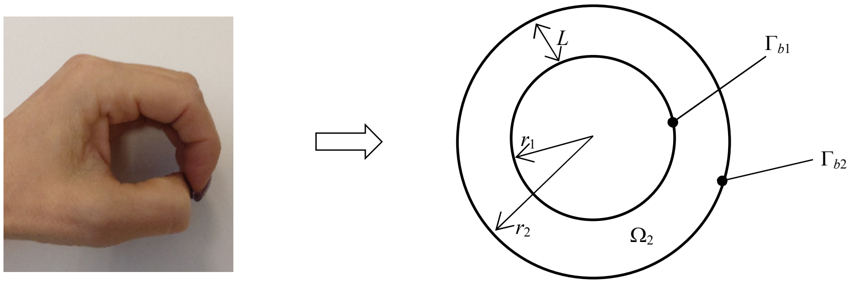

Fig. 3. Sketch of a free hand

The heat conduction equation in the region Ω_2 takes the following form:

$$\lambda_2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T_{init,2}\left(r\right)}{\partial r}\right) + Q_{met} + Q_{perf}\left(r\right) = 0 \tag{14}$$

where here $Q_{perf}\left(r\right) = c_{bl}\rho_{bl}G_{bl}\left(T_{bl} - T_2\left(r\right)\right)$.

The boundary conditions are:

- on the boundary Γ_{b1}, i.e. for $r = r_1$:

$$\lambda_2 \left.\frac{\partial T_{init,2}\left(r\right)}{\partial r}\right|_{r=r_1} = \alpha_2 \left(T_{init,2}\left(r\right)\Big|_{r=r_1} - T_{amb2}\right), \tag{15}$$

- on the boundary Γ_{b2}, i.e. for $r = r_2$:

$$-\lambda_2 \left.\frac{\partial T_{init,2}\left(r\right)}{\partial r}\right|_{r=r_2} = \alpha_2 \left(T_{init,2}\left(r\right)\Big|_{r=r_2} - T_{amb2}\right) \tag{16}$$

where T_{amb2} is the temperature of the cold environment, and α_2 is the heat transfer coefficient related to this environment.

2.3. Determination of the degree of the tissue burn

The tissue burns are classified as first-, second-, or third-degree. One of the methods that allows determining the degree of the tissue burn is based on the value of the Arrhenius integral, which is defined as follows:

$$I(r,t)=\int_0^t P\cdot\exp\left(-\frac{\Delta E}{R_g\cdot\left(T_2(r,\tau)+273\right)}\right)d\tau \tag{17}$$

where P is the pre-exponential factor, ΔE is the activation energy, and R_g is the universal gas constant. The values of the integral I, given by the equation (17), can be used to analyse reactions occurring during heating and to determine the probability of the degree of tissue burn. For example, the value of the integral I in the range $[0.53, 1]$ denotes the probability of first-degree burn in the range $[1, 10000]$ – the probability of second-degree burn, and above the value 10000 – third-degree burn.

3. Numerical examples

The solution of the equations of the mathematical model presented in the previous section allows determining the temperature distribution in the case when the hand is squeezing a pipe through which hot water flows. The numerical calculations have been performed using the finite volume method [14, 15].

In computations, we assume the following values of hand parameters: the thickness of a hand $L=r_2-r_1=20$ mm; $c_2=3421$ J/(kg°C); $\rho_2=1090\ \mathrm{kg/m^3}$; $\lambda_2=0.48$ W/(m°C); $Q_{met}=550\ \mathrm{W/m^3}$; $c_{bl}\rho_{bl}=3770\cdot1060\ \mathrm{J/(m^3\,°C)}$; $T_{bl}=37$°C; $G_{bl}=0.0005\ \mathrm{m^3_{blood}/(s\cdot m^3_{tissue})}$. Moreover, the remaining geometrical and physical data have been assumed as: $\alpha=10\ \mathrm{W/(m^2\,°C)}$; $T_{amb1}=20$°C; $\alpha_2=20\ \mathrm{W/(m^2\,°C)}$; $T_{amb2}=5$°C; $P=3.1\cdot10^{98}$ 1/s; $\Delta E=6.27\cdot10^5$ J/mol; $R_g=8.314$ J/(mol°C).

In the first example, the pipe is made of steel (typical ½ inch pipe) and is specified by: $r_1=10.7$ mm; $r_1-r_0=2.6$ mm; $c_1=486$ J/(kg°C); $\rho_1=7753\ \mathrm{kg/m^3}$; $\lambda_1=36$ W/(m°C). In Figure 4, plots of the function $T(r,t)$ illustrating the temperature distributions in the cross-section of the considered water, pipe, and hand for different values of the temperature of the water $T_{water}=60$°C; 70°C; 80°C; 90°C are presented.

In the second example, the 20 mm diameter pipe is made of PP-R polypropylene. Values of pipe parameters are the following: $r_1=10$ mm; $r_1-r_0=1.9$ mm; $c_1=2000$ J/(kg°C); $\rho_1=895\ \mathrm{kg/m^3}$; $\lambda_1=0.24$ W/(m°C). As in the previous example, plots of the functions illustrating the temperature distributions in the cross-section of the considered structure are presented for the same values of water temperature $T_{water}=60$°C; 70°C; 80°C; 90°C (Fig. 5).

In both examples, we can see, as we expected, that warming up the hands by squeezing a pipe through which hot water flows is effective, although it must be done very carefully. The hotter water and the longer time of holding the pipe may result in a greater degree of tissue burns. We can also see that, in this respect, it is safer to warm the hand by holding a PP-R (polypropylene) pipe than a steel pipe.

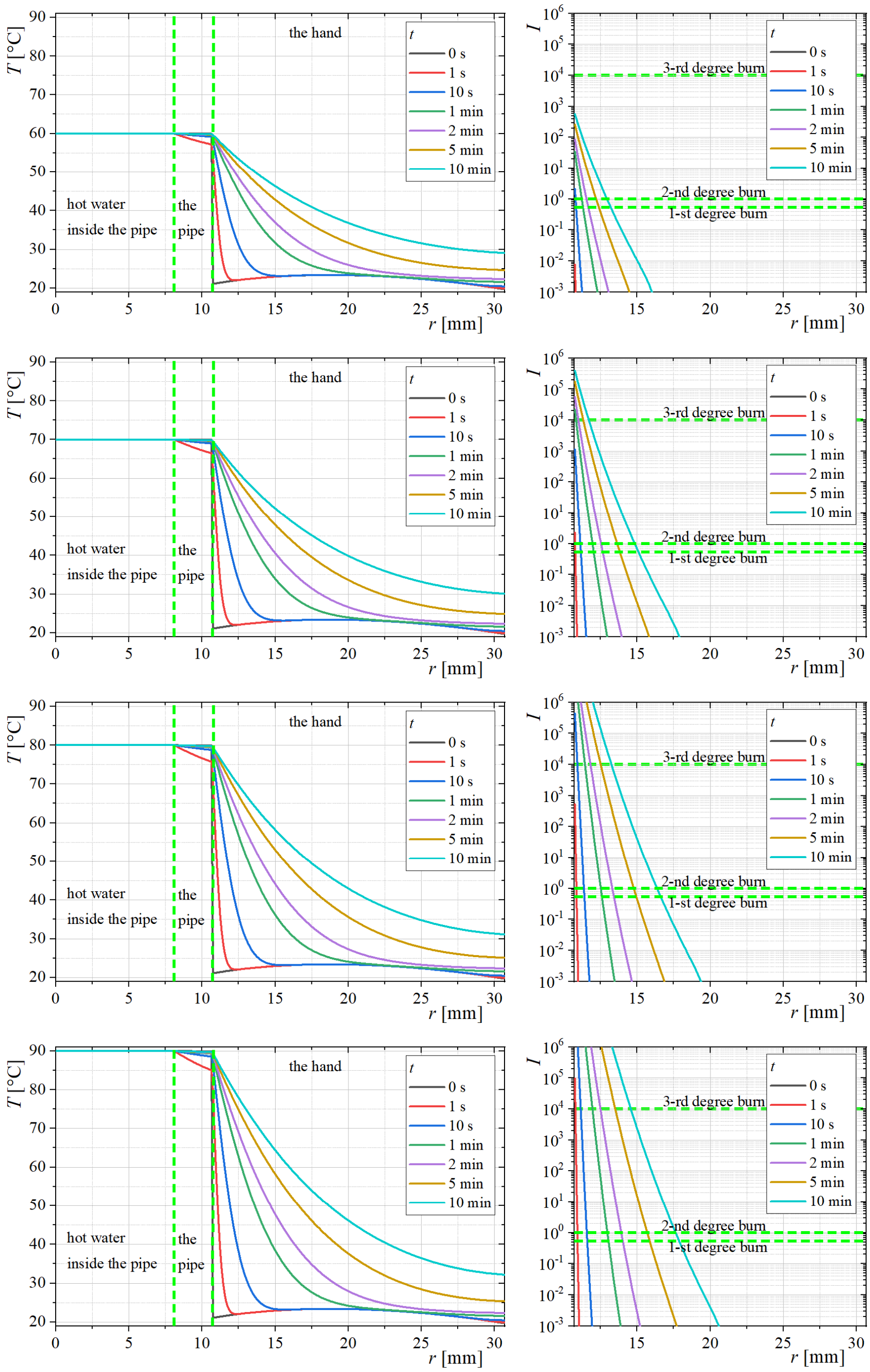

Fig. 4. Distributions of the temperature in the cross section of the structure water – steel pipe – hand for T_{water} = 60 °C; 70 °C; 80 °C; 90 °C

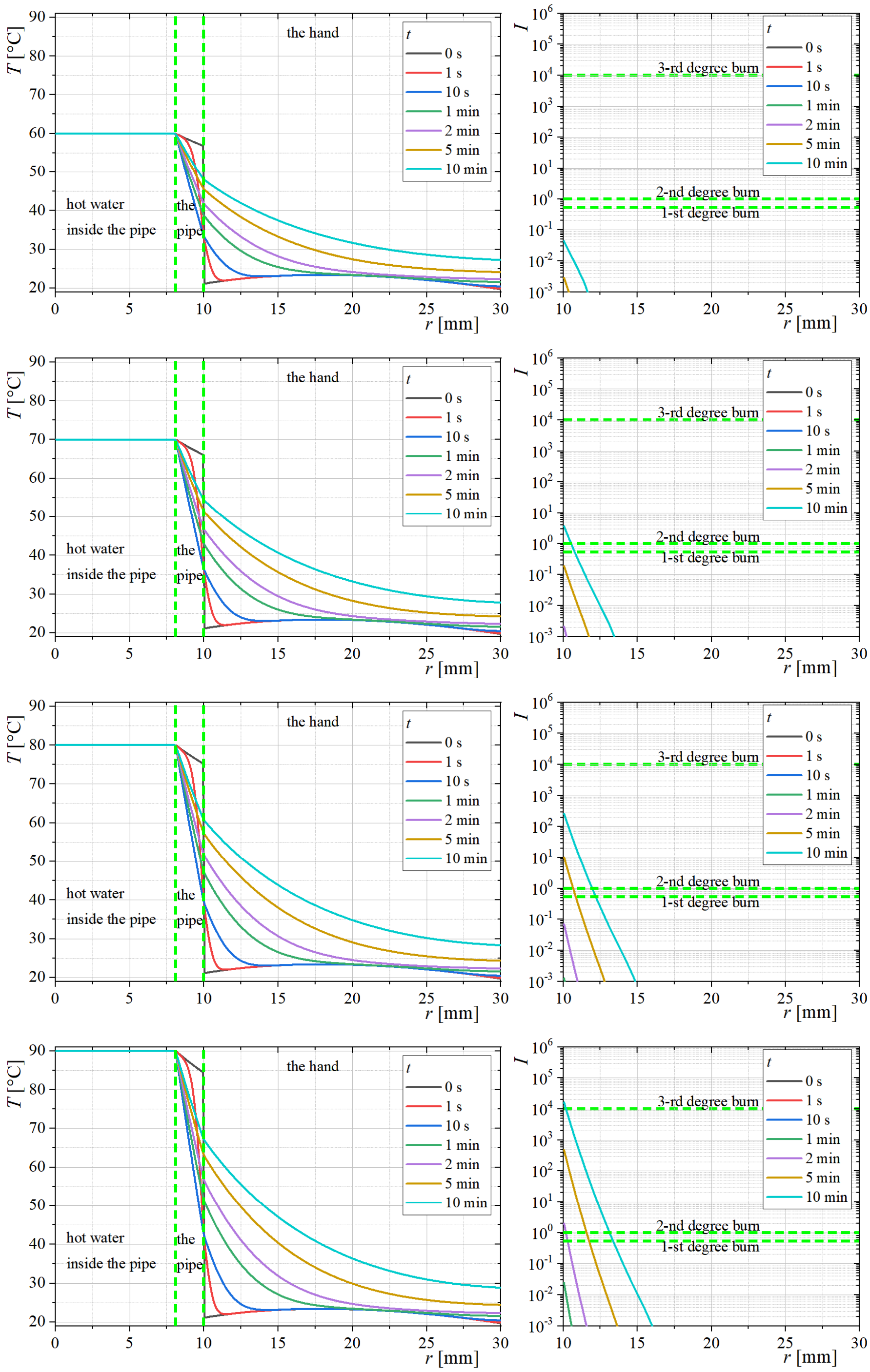

Fig. 5. Distributions of the temperature in the cross section of the structure water – PP-R pipe – hand for T_{water} = 60 °C; 70 °C; 80 °C; 90 °C

4. Conclusions

In this work, the issue of warming up a cold (or frozen) hand that is squeezing a pipe through which hot water/liquid flows was considered. Often, such hand warming practices are performed by people who spend a long time in a cold environment, and their hands are not protected from the cold, e.g. with gloves. From a medical point of view, such hand warming is effective under certain conditions. It turns out that touching/squeezing a hot water pipe for too long can cause skin burns and thermal destruction of internal tissues. By analysing the sample results, it is possible to estimate different degrees of burns depending on the temperature of the flowing hot water, the material of the pipe and the heating time. It turns out that a pipe made of PP-R polypropylene (characterized by a low thermal conductivity coefficient) is more suitable for safe hand warming than pipes made of metals or alloys, where heat is intensively transferred from hot water to the hand through this material. Often, when a person feels thermal discomfort caused by long-term contact of the hand with a hot pipe, the hand is involuntarily released from such contact, often even after squeezing the very hot pipe for a few seconds, in order to prevent the hand from burning. The presented calculation results, lasting even several dozen minutes of contact, should be treated as illustrative only, and it is not recommended to test them on your own experimental experiences.

The mathematical model considered is quite simple and concerned a one-dimensional axisymmetric task in cylindrical coordinates, i.e. in fact it described the heat flow in the cross-section in the central part of the hand squeezing the pipe. In the future, the model can be expanded to include a two-dimensional axisymmetric task, and other simulation cases can be considered, e.g. with a layer of thermal insulation of the hand (i.e. using a glove). Estimating the degree of burns using the burn integral according to the Arrhenius model should be treated with a certain level of uncertainty, because each person has different properties of the skin or subcutaneous tissues and a different level of metabolism or blood perfusion. Hence, the obtained burn integral values should be treated as estimates.

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