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DEVELOPMENT OF THE STATISTICAL MODEL FAILURE OF ORTHOTROPIC COMPOSITE MATERIALS

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Abstract. A composite plate (matrix and reinforcing elements) under conditions of plane deformation is considered. According to the elastic properties, the material of the plate is considered orthotropic with uniformly distributed defects-cracks that do not interact with each other. The geometric characteristics of defects are statistically independent random variables – the half-length and the orientation angle between the defect line and the axis of orthotropy with a larger Young's modulus. The ratio for the failure loading integral probability distribution function of the composite was obtained. The dependencies of the researched composite probability of failure (reliability) for the different number of cracks (plate sizes), different types of loading and various values of the exponential distribution parameter are calculated and investigated graphically.

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Keywords: *stochasticity, probability of failure, orthotropic composite material, failure loading, integral probability distribution function*

1. Introduction

The problem of the composite materials strength, which occupy an important place in the design of structural elements, is an urgent task. Taking into account the randomness and stochasticity (certain probability distribution) of their structural descriptive parameters are determining aspects for assessing their strength and reliability. The complex application of deterministic solutions to the problems of brittle fracture mechanics and statistical-probabilistic methods allows this assessment to be carried out qualitatively. The study of this direction was carried out in the works of a number of authors. The article [1] presents a methodology for reliability assessment of composite members based on appropriate limit state functions derived according to fundamental failure criteria, applicable to composite materials. The methodology that is proposed employs a stochastic response surface method which combines in discrete steps modelling, numerical simulations and analytical probabilistic assessment techniques. In paper [2], using Weibull distribution, the welded joint reliability can be obtained. The probability distribution according to which the joint will fail is estimated. In study [3], experiments are carried out to determine the tensile strength of laminates, for three different orientations of glass/epoxy and carbon/epoxy composites. Using two-parameter Weibull distribution, the theoretical tensile strength values are determined for glass fiber reinforced polymer and carbon fiber reinforced polymer composites for different strain rates. The effect of wet-dry cycling times on the failure modes, tensile strength, and the probability distribution of different fiber reinforced polymer composite specimens were investigated [4]. According to the experimental results, a probability analysis was conducted on the degradation of tensile strength. The work [5] presents an extension of a previously developed numerical framework utilizing discontinuous solid-shell elements and enriched cohesive elements for the simulation of damage growth of open-hole laminates under compression. The role of microstructural bridging on the fracture toughness of composite materials was investigated in [6]. To achieve this, a new computational framework is presented that integrates phase field fracture and cohesive zone models to simulate fibre breakage, matrix cracking and fibre-matrix debonding. The calculation of the failure probability in the carbon fibre/epoxy-based composite material was carried out [7]. The effective orthotropic properties of the composite for various fibre-volume fractions have been numerically computed by the homogenisation method using periodic boundary conditions. In paper [8], an adaptive multi-fidelity modelling approach is proposed, wherein actively damaging areas are modelled with high-fidelity three-dimensional brick elements and discrete cracks, while dormant and inactive sites are modelled with lower-fidelity shell elements and smeared cracks. The transition criteria between the two levels of modelling are studied in order to preserve as much fidelity to the physics as necessary while improving computational efficiency.

The purpose of this study is to develop a statistical model of failure (reliability assessment) of orthotropic composite materials under conditions of plane deformation, taking into account the structural heterogeneity of the material.

2. Formulation of the problem

We consider a plate made of a composite material consisting of a matrix and reinforcing elements. According to its elastic properties, such a material can be considered as orthotropic. The plate is under the conditions of action of a loading P and Q $(Q = \eta P)$ uniformly distributed along the edges (conditions of plane deformation) (Fig. 1). The coordinate axes ox_1 and oy_1 correspond to the main axes of orthotropy, E_1 and E_2 are the Young's modulus for tension-compression along the axis ox_1 and the axis oy_1, respectively $(E_1 > E_2)$.

It is known that the structural heterogeneity of the material is an important parameter that determines the failure nature around the crack, in addition to the strength properties. Therefore, when formulating the composite materials failure criterion under the conditions of a complex stress state, taking this heterogeneity into account is an important task. Among various defects, cracks play a special role, because they cause a significant concentration of stresses in a deformed body. The development of such defects leads to local or complete failure of the composite. In the material under study, there are evenly distributed defects-cracks that do not interact with each other. The geometric characteristics of defects are their half-length l and the orientation angle α between the defect line and the orthotropy axis ox_1 (Fig. 1). They are statistically independent random variables. The defect of the structure is characterized by the joint probability distribution density $f(\alpha, l)$ and the integral probability distribution function $F(\alpha, l)$, which are set on the basis of structural analysis or a priori general considerations.

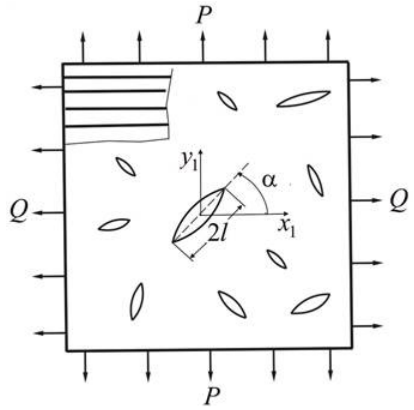

Fig. 1. Model of orthotropic composite material with random defects

3. Probability distribution densities of defects geometric characteristics

We accept the hypothesis about the most probable orientation of cracks in the direction of the axis ox_1 (reinforcement direction, higher Young's modulus E_1), which is confirmed by experimental studies [9]. We enter the Young's moduli ratio $\lambda = E_1/E_2 > 1$. According to the accepted hypothesis, we choose the probability distribution density of a random variable α in the form [10]

$$f(\alpha) = \frac{\lambda^{3/2}}{\pi(\lambda^3 \sin^2\alpha + \cos^2\alpha)}. \quad (1)$$

When $\lambda = 1$ we get a uniform distribution that corresponds to the isotropic material structure defectiveness.

Probability distribution density graphs $f(\alpha)$ (1) for different Young's moduli ratio (various values of the parameter λ) (Fig. 2) were presented in [10]. In particular, the value $\lambda = 3.2$ corresponds to the specimen from the epoxy phenolic fiberglass EF 32-301 on the cord glass fiber TSC-VM-1-78 [10], for which the crack propagation along the main direction of reinforcement has been experimentally confirmed.

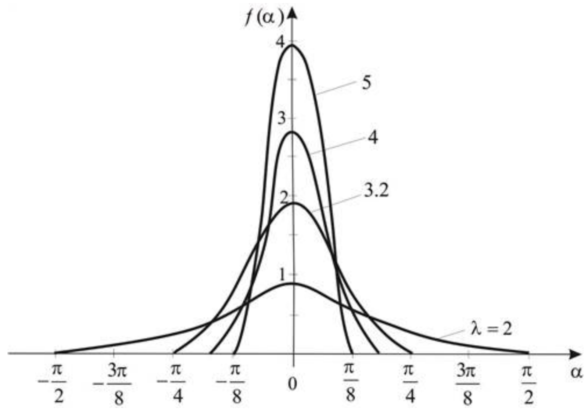

Fig. 2. Probability distribution density $f(\alpha)$ for different values of the parameter λ

We will assume that the random variable l varies in a certain interval $0 \le l \le d$, where d is the finite structural characteristic of the material. In paper [11], a failure criterion that takes into account the micro heterogeneity of the material is proposed. The initial direction of crack propagation in the general case of plane tension-compression of a micro heterogeneous brittle body is established based on the well-known concept of macro stresses (stresses averaged over some area).

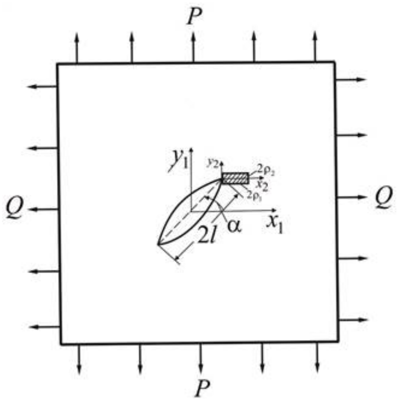

Fig. 3. A structural element centered at the top of the defect

We introduce a structural element with the center at the top of the defect and dimensions $2\rho_1$ and $2\rho_2$ [12] (Fig. 3). The parameters ρ_1 and ρ_2 depend on the size, type and density of the material structure heterogeneity distribution. The size of the structural element ρ_1 is small in comparison with the magnitude l. The value is ρ_2 equal to the distance between the centers of adjacent reinforcing fibers $(\rho_2 \ll \rho_1)$.

Let's introduce a random variable $L = \rho_1 / l$ that changes over an interval: $0 \le L < \infty$, where ρ is a fixed parameter. This assumption makes it possible to simplify the material model and mathematical calculations.

In [10], the probability distribution density of a random variable L was chosen in the form of a power distribution

$$f(L) = \frac{(s-1)\,a^{s-1}}{(L+a)^s},\ s > 1,\ a > 0\,. \tag{2}$$

Distribution (2) is a two-parameter statistical model, where s is a form parameter, a is a scale parameter.

In this study, we choose the probability distribution density of the random variable L, in accordance with the results of the strength statistical theory development [13, 14], in the form of an exponential law

$$f(L) = e^{-L/h}/h\,. \tag{3}$$

Here h is a distribution parameter having the dimension of a random variable L. According to the physical meaning of law (3), the probability of value L occurrence decreases with its increasing.

The parameter L integral probability distribution function will be written as follows:

$$F(L) = 1 - e^{-L/h}\,.$$

Graphs of the parameter L probability distribution density in the case of power [10] and exponential distributions are shown in Figure 4.

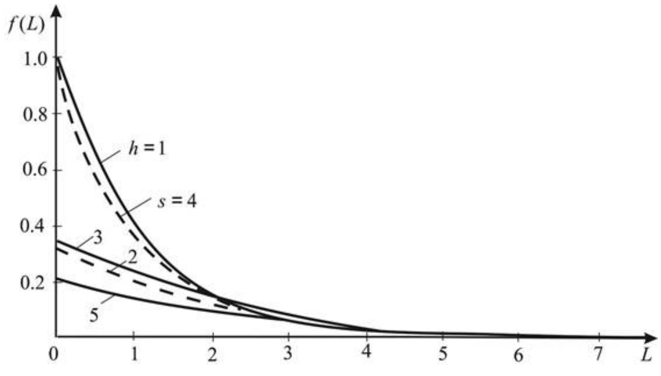

Fig. 4. Probability distribution density $f(L)$ (solid for the exponential law, dashed for the power law at $a = 3$)

As the random variable L increases, the probability distribution density curves $f(L)$ asymptotically approach zero. There is a certain range of the parameter L in which a significant change in the value of the probability distribution density occurs. For certain parameter values a, s and h, the curves of power and exponential laws differ little from each other.

Joint probability distribution density of statistically independent random variables α and L according to (1), (3) has the form

$$f(\alpha,L)=\frac{\lambda^{3/2}e^{-L/h}}{\pi h(\lambda^3\sin^2\alpha+\cos^2\alpha)}. \tag{4}$$

Graphs of the joint probability distribution density $f(\alpha,L)$ (4) are shown in Figure 5.

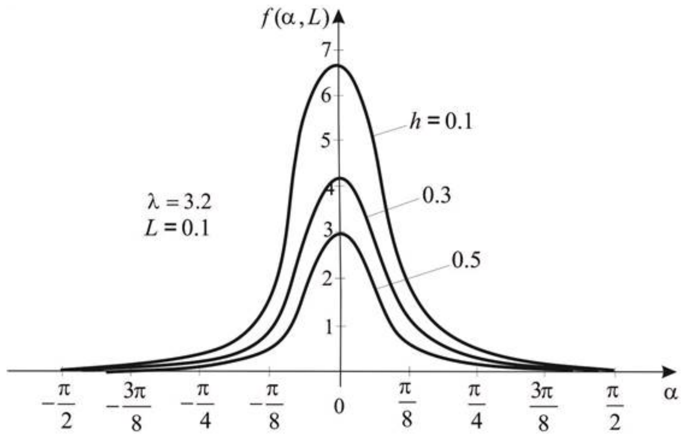

Fig. 5. Joint probability distribution density $f(\alpha,L)$ for different parameter values h

The graphs $f(\alpha,L)$ are symmetrical about the y-axis. As the parameter h changes, the shape of the curve $f(\alpha,L)$ changes. An increase in the value of the parameter h leads to a decrease in the maximum of the joint probability distribution density.

4. Failure loading integral probability distribution function

According to the methodology [15] and expression (4), the failure loading integral probability distribution function for a composite element with one defect is written as follows:

$$F_1(P,\eta)=\frac{\lambda^{3/2}}{\pi h}\iint_{\Omega_{\alpha,L}}\frac{e^{-L(\rho_1,l,\alpha,P,\eta)/h}}{\lambda^3\sin^2\alpha+\cos^2\alpha}d\alpha\, dL,\ 0\le P\le P_{\max}. \tag{5}$$

Here $\Omega_{\alpha,L}$ is the integration area: $-\pi/2 \le \alpha \le \pi/2$, $0 \le L < \infty$, the value $L(\rho_1, l, \alpha, P, \eta)$ is determined from the failure criterion [10], which is expressed through the components of macroscopic stresses $[\sigma_{ij}]$ $(i = 1,2;\ j = 1,2)$

$$[\sigma_{11}]\sin^2\alpha + [\sigma_{22}]\cos^2\alpha + [\sigma_{12}]\sin 2\alpha = \sigma_{cr}, \qquad (6)$$

where σ_{cr} is the strength of the composite.

For an arbitrarily oriented crack, the macroscopic stresses $[\sigma_{ij}]$ are written as

$$[\sigma_{ij}] = \Gamma_{ij} + \sum_{m=1}^{3} L^{0.5m-1} \operatorname{Re} \sum_{k=1}^{2} s_{ij}^{(k)} \left(\frac{1}{3} A_k^{(1)} C_k^{(1)} + A_k^{(2)} + \frac{1}{15} A_k^{(3)} C_k^{(3)} \right); \qquad (7)$$

where the following notations are introduced:

$$\begin{aligned}
&\Gamma_{11} = P(\sin^2\alpha + \eta\cos^2\alpha),\ \Gamma_{22} = P(\cos^2\alpha + \eta\sin^2\alpha),\\
&\Gamma_{12} = 0.5P(1-\eta)\sin 2\alpha; \quad s_{11}^{(k)} = \mu_k^2,\ s_{12}^{(k)} = -\mu_k,\ s_{22}^{(k)} = 1;\\
&A_k^{(m)} = (-1)^{m+1} \frac{(-1)^{k+1} m}{2(\mu_2 - \mu_1)} \left(t_k D^{(1)} + D^{(2)} \right),\ t_1 = \mu_2,\ t_2 = \mu_1;\\
&D^{(1)} = P(\cos^2\alpha + \eta\sin^2\alpha),\ D^{(2)} = 0.5(1-\eta)\sin 2\alpha;\\
&C_k^{(m)} = \frac{2^{0.5m+1}}{a_k b_k} \frac{\rho_2}{\rho_1} \left(\left(2a_k + b_k \frac{\rho_2}{\rho_1} \right)^{0.5m+1} - \left(b_k \frac{\rho_2}{\rho_1} \right)^{0.5m+1} - \left(2a_k \right)^{0.5m+1} \right);\\
&a_k = \cos\alpha - \mu_k \sin\alpha,\ b_k = \sin\alpha + \mu_k \cos\alpha.
\end{aligned} \qquad (8)$$

Here μ_k are complex parameters, which are determined from the characteristic equation [16]

$$\frac{\mu_k^4}{E_1} + \left(\frac{1}{G_{12}} - \frac{\nu_{12}}{E_1} \right) \mu_k^2 + \frac{1}{E_2} = 0,\ k = 1,2, \qquad (9)$$

where ν_{12} is Poisson's coefficient, G_{12} is the shear modulus.

Taking into account the elastic material characteristics $E_1 = 50\,100$ MPa, $E_2 = 15\,600$ MPa, $\nu_{12} = 0.25$ [9], were obtained [10] the following solutions of equation (9): $\mu_1 = 2.7874i$, $\mu_2 = 0.6429i$.

We reduce the double integral (5) to the corresponding repeated integral, and taking into account that the value of the integral probability distribution function of the parameter L belongs to the segment [0;1], we obtain

$$F_1(P,\eta) = \frac{\lambda^{3/2}}{\pi h} \int_{-\pi/2}^{\pi/2} \frac{e^{-L(\rho_1, l, \alpha, P, \eta)/h}}{\lambda^3 \sin^2\alpha + \cos^2\alpha} d\alpha,\ 0 \le P \le P_{\max}. \qquad (10)$$

5. The probability of failure of orthotropic composite material

A plate containing N defects probability of failure (reliability) we find by the formula [15], which is based on the hypothesis on the weakest link

$$P_f = 1 - \left(1 - F_1(P,\eta)\right)^N, \ 0 \le P \le P_{\max}. \tag{11}$$

Taking into account expressions (10), (11), the orthotropic composite material plate with N cracks probability of failure for biaxial tension-compression

$$P_f = 1 - \left(1 - \frac{\lambda^{3/2}}{\pi h} \int_{-\pi/2}^{\pi/2} \frac{e^{-L(\rho_1,l,\alpha,P,\eta)/h}}{\lambda^3 \sin^2\alpha + \cos^2\alpha} d\alpha \right)^N, \ 0 \le P \le P_{\max}. \tag{12}$$

According to the formula (12), taking into account failure criterion (6)-(8), the dependencies of the researched composite probability of failure P_f for the different number of cracks N, different types of loading (parameter η) and various values of the distribution parameter h are calculated. The corresponding diagrams are shown in Figures 6 and 7.

In Figure 6, the diagrams are shown for different types of composite stressed state at $h = 1$ for uniaxial tension $(\eta = 0)$, for equal biaxial tension $(\eta = 1)$ and for tension-compression $(\eta = -1)$. Diagrams are constructed for different number of cracks.

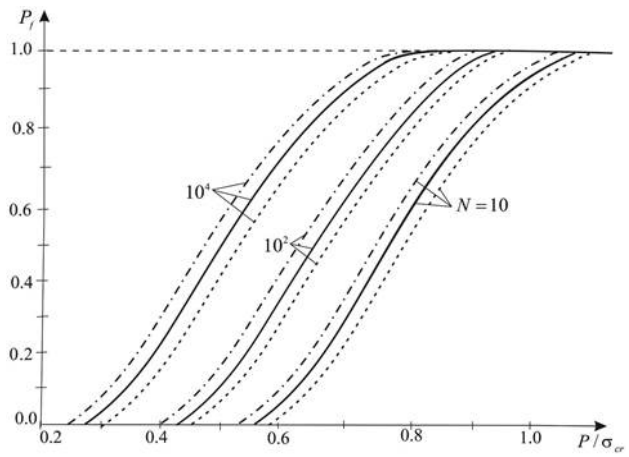

Fig. 6. Probability of failure for various types of stress state (solid for $\eta = 0$, dashed for $\eta = 1$, dotted dashed for $\eta = -1$)

Figure 7 shows the dependence of the probability of failure in the fixed dimensions of the composite ($N = 60$) on the different material structural heterogeneity and for the different types of loading.

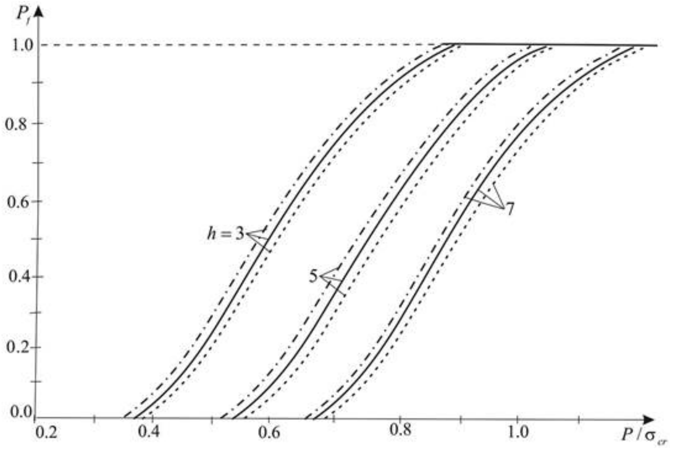

Fig. 7. Probability of failure of the composite for different values of the parameter h (solid for $\eta = 0$, dashed for $\eta = 1$, dotted dashed for $\eta = -1$)

6. Conclusions

In Figure 6, we observe the dependence of the probability of failure P_f on the type of stress state (influence of the parameter η). For a fixed loading, we get the highest value P_f for tension-compression $(\eta = -1)$, as well as its increase with an increase in the number of defects N. A certain range of loading corresponds to a low probability of failure.

Figure 7 shows the dependence of the probability of failure in the fixed dimensions of the composite $(N = 60)$ for different material structural inhomogeneity (parameter h) at different types of loading. We obtain the pattern of the probability of failure decreasing for a fixed loading when the material structure goes to the homogeneous with an increase in the parameter h. This pattern depends on the type of stressed state (on η).

According to the physical meaning of law (1), for the proposed material model, the predominant orientation of defects is in the direction of reinforcement. Therefore, it can be concluded that the lowest probability of failure of the studied material under equal biaxial tension, i.e., the highest reliability, is a consequence of the action of the loading Q, which closes the cracks and increases of the material strength. Therefore, we observe the influence of the material structure orthotropy.

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