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CONDITIONAL STABILITY ESTIMATE FOR AN ILL-POSED ELLIPTIC EQUATION BY USING NONLOCAL CONDITIONS

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Abstract. We consider an ill-posed linear homogeneous fourth-order elliptic equation. We show that the problem is ill-posed in the sense of Hadamard, i.e., the solution does not depend continuously on the given data. We propose a regularization method via nonlocal conditions and under some a priori bound assumptions different estimates for the regularized solution are obtained. Numerical examples for a rectangle domain show the effectiveness of the new method in providing highly accurate numerical solutions as the noise level tends to zero.

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1. Introduction

Let $\Omega = \{(x,y):\ 0 < x < \pi,\ 0 < y < \ell\}$. This paper is concerned with the study of the ill-posedness of homogeneous biharmonic equation

$$\Delta^2 v = v_{yyyy}(x,y) + 2v_{yyxx}(x,y) + v_{xxxx}(x,y) = 0, \quad (x,y) \in \Omega, \tag{1}$$

where v is the elastic displacement, $\Delta^2 v = \Delta(\Delta v)$.

The paper [1] is the first one where the conditional stability estimate in a rectangular domain was proved for the homogeneous biharmonic equations with different Cauchy data (see Theorem 7.1 in Section 7). However, they did not present the error estimates. We consider the model given by [1] in which the boundary conditions are given on all edges of the domain as follows

$$v(0,y) = 0,\ \Delta v(0,y) = 0, \quad v(\pi,y) = 0,\ \Delta v(\pi,y) = 0, \tag{2}$$

$$v(x,0) = f(x),\ \frac{\partial v}{\partial y}(x,0) = g(x), \quad 0 \le x \le \pi, \tag{3}$$

$$\Delta v(x,\ell)=h(x),\ \frac{\partial \Delta v}{\partial y}(x,\ell)=l(x), \quad 0\leq x\leq \pi, \tag{4}$$

where $\ell>0$ is a real constant and $f,g,h,l\in L^2(0,\pi)$ are known. For easy reading we denote by $(\mathscr{P})$ the problem (1)-(4). In fact, it can be proven (see, Section 2) that this problem has a unique solution which does not depend continuously on the given noisy data, where any small change of the data may cause dramatically large error in the solution. Well-posed Biharmonic problems are useful in mechanics to describe some basic equations in plane elasticity. In [2], the equation (1) was used to describe the stress of a plate in linear elasticity. There have been many results, and various methods have also been presented from both mathematical and numerical points of view, such as the motion of fluids [3], free boundary problems [4], and the problems related to blending surface [5]. For a more elaborate history of the biharmonic problem and the relation with elasticity, see the survey of Meleshko [6] and the monograph [7].

In the literature, many numerical methods have been developed for solving the well-posed biharmonic problem. We present a selective review of the main and the most recent methods. These methods include: fundamental solutions [8,9], the iterative solution of finite difference approximation [10], the iterative method of Kozlov was given in [11], the finite element treatment was used in [12] and finite volume schemes were used in [13], the method of approximate fundamental solutions in [14] and the boundary element method in [15].

Our aim is to give a regularization method, based on the idea of replacing two local boundary conditions with two nonlocal ones from observed boundary data $f(x),g(x),h(x)$ and $l(x)$. We then propose a semi-discrete finite difference method to verify the stability of our proposed regularization method. To the best of our knowledge, the first applications of this idea to the topic of ill-posed biharmonic equations were done in paper of Benrabah and Boussetila [16]. It should be noticed that the problem $(\mathscr{P})$ with $f(x)=h(x)=0$, has been studied in [17]. For readers interested in the ill-posedness of elliptical PDEs in parameter estimation, i.e. inverse problems, or in non-local boundary conditions, one can mention the following recent papers: [18,19]. The rest of the paper is organized as follows: In Section 2, notations and some definitions are given, and we also consider the formulation of a solution of problem $(\mathscr{P})$ and its ill-posed property. In Section 3, the regularization method will be given. We present numerical experiments in section 4. Finally, conclusions are given in the last section.

2. Ill-posedness of the problem

Definition 1 We call a function $v\in C^4(\Omega)\cap C^3(\overline{\Omega})$ satisfying equation (1) and the boundary conditions (2)-(4) a classical solution. □

The equation (1) can be written as $\left(\frac{\partial^2}{\partial y^2} - \mathbb{A}\right)^2 v(x,y) = 0$, and it is easy to check that the operator $\mathbb{A} = -\frac{\partial^2}{\partial x^2}$ with $\mathscr{D}(\mathbb{A}) =\subset L^2(0,\pi)$, is positive, self-adjoint with compact resolvent. The eigenvalues $\lambda_k = k^2, (k \in \mathbb{N}^*))$ and the corresponding eigenvectors $\psi_k = \sqrt{\frac{2}{\pi}} \sin(kx)$, $k \in \mathbb{N}^*$, which form an orthonormal basis in $L^2(0,\pi)$.

Definition 2 The abstract Gevrey class of functions of order $p > 0$ and index q, defined by

$$\mathscr{G}_{p,q} = \left\{ \varphi \in L^2(0,\pi) : \|\varphi\|^2_{p,q} = \sum_{k=1}^{\infty} \frac{e^{2pk\ell}}{k^q} c_k^2(\varphi) < +\infty \right\} \quad p \geq 0, q \in \mathbb{R}, \tag{5}$$

where $c_k(\varphi) = \langle \varphi, \psi_k \rangle_{L^2(0,\pi)} = \int_0^\pi \varphi(x)\psi_k(x)dx = \sqrt{\frac{2}{\pi}} \int_0^\pi \varphi(x)\sin(kx)dx$ is the Fourier coefficient of the function φ. □

The norm in $\mathscr{G}_{p,q}$ and $L^2(0,\pi)$ will be denoted respectively by $\|.\|_{p,q}$ and $\|.\|$. For simplicity, we will analyze our problem with the simplest choice $l(x) = 0$. The generalization of the following convergence analysis to the case $l(x) \neq 0$ requires some nontrivial extra efforts since the solution has a different expression.

Let E_{data} be the set of exact data, i.e., $E_{\text{data}} = \{f, g, h\}$ and we also denote the measured data by $E^\delta_{\text{data}} = \left\{f^\delta, g^\delta, h^\delta\right\}$. The solution to problem $(\mathscr{P})$ can be represented in the form of an expansion in the orthogonal series

$$v(x,y) = \sum_{k=1}^{\infty} \psi_k(x) w_k(y) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} w_k(y)\sin(kx), \tag{6}$$

By substituting (6) in the equation (1) and in the boundary conditions (2)-(4), we obtain the solution

$$\begin{aligned} v(x,y) &= \sum_{k=1}^{\infty} \left\{ \left(\frac{1}{2k^2} \sinh(ky)\sinh(k\ell) + \frac{y}{2k} \sinh(k(y-\ell)) \right) c_k(h) \right. \\ &+ \left. \cosh(ky) c_k(f) + \frac{\sinh(ky)}{k} c_k(g) \right\} \psi_k(x) \end{aligned} \tag{7}$$

of the problem $(\mathscr{P})$; where $c_k(f), c_k(g)$ and $c_k(h)$ are the Fourier coefficients of the expansion according to the orthonormal basis $\{\psi_k(x)\}_{k=1}^{\infty}$ of the functions f, g and h respectively. By observing (7), one may recognize that the growth of $v(x,y)$ is of exponential order. Thus, the solution will be destroyed at high frequencies. Formally, we have the following example regarding the ill-posed structure of the problem $(\mathscr{P})$. For example, if we take $g(x) = h(x) = 0$, and $f(x) = \sqrt{\frac{2}{\pi}} \frac{\sin(kx)}{k}$, then the solution is given by $v(x,y) = \sqrt{\frac{2}{\pi}} \cosh(ky) \frac{\sin(kx)}{k}$, Since $\lim_{k\to\infty}\|f\|^2_{L^2(0,\pi)} = \lim_{k\to\infty}\int_0^\pi \left|\sqrt{\frac{2}{\pi}}\frac{\sin(kx)}{k}\right|^2 dx = 0$, and for fixed $y > 0$, we have

$$\lim_{k\to\infty}\|v(.,y)\|^2_{L^2(0,\pi)} = \lim_{k\to\infty}\frac{\cosh^2(ky)}{k^2} = +\infty.$$

Consequently, the considered problem $(\mathscr{P})$ is ill-posed in the sense of Hadamard [20] in the $L^2(0,\pi)$-norm.

3. Regularization method

As previously mentioned, the ill-posedness of the investigated problem is mainly caused by the exponential growth of the hyperbolic functions $\cosh(ky)$ and $\sinh(ky)$. Thus, to obtain a stable approximation of the original problem, one must control the growth of the component e^{ky} as $k\to\infty$. In this direction, the idea arising here is to replace the boundary condition $v(x,0) = f(x)$ in (3) and the boundary condition $\Delta v(x,\ell) = h(x)$ in (4) with

$$\Theta_v^\alpha(f^\delta) = v(x,0) + \alpha v(x,\ell) = f^\delta(x), \tag{8}$$

$$\Theta_{\Delta v}^\alpha(h^\delta) = \Delta v(x,\ell) + \alpha\Delta v(x,0) = h^\delta(x), \tag{9}$$

respectively, where $\alpha > 0$ is the regularization parameter.
We obtain a nonlocal problem $\left(\mathscr{P}_\alpha^\delta\right)$ with the new nonlocal conditions given by (8) and (9) where the measured data $E^\delta_{\text{data}} \in \left(L^2(0,\pi)\right)^3$, satisfies

$$\|f^\delta - f\|_{L^2(0,\pi)} \le \delta, \quad \|g^\delta - g\|_{L^2(0,\pi)} \le \delta, \quad \|h^\delta - h\|_{L^2(0,\pi)} \le \delta, \tag{10}$$

in which the constant $\delta > 0$ is the noise level. The following technical lemma play the key role in our analysis and calculations.

Lemma 1 *([16]) Let* $[1,+\infty[\ni z\mapsto \mathscr{R}_r(z) = \dfrac{1}{\alpha z^r + 2e^{-z\ell}}$, *where* $\alpha > 0,\ \ell > 0,$ *and* $r > 1$. *Then one has*

$$\mathscr{R}_r(z) \le \frac{1}{\alpha}\left(\frac{\ell_1}{\ln(\ell_2(1/\alpha))}\right)^r, \tag{11}$$

where $\ell_1 = r\ell$, $\ell_2 = 2(\ell)^r/r$, *and* $0 < \alpha < 2\ell^r/(re)$.
If $r = 1$, *the function* $z\mapsto\mathscr{R}_1(z)$ *can be also estimated as follow*

$$\mathscr{R}_1(z) \le \frac{\ell}{\alpha\ln(2\ell/\alpha)}, \tag{12}$$

for $0 < \alpha < 2\ell/e$. □

For $k \geq 1$, we introduce the functions

$$\cosh_\alpha(k) = (1+\alpha\cosh(k\ell)) \leq \cosh_\alpha^2(k), \tag{13}$$

$$\mathscr{V}_\alpha(k,y) = \frac{\sinh(ky)}{\cosh_\alpha(k)} \leq \frac{1}{\alpha}, \qquad \mathscr{W}_\alpha(k,y) = \frac{\cosh(ky)}{\cosh_\alpha(k)} \leq \frac{1}{\alpha}. \tag{14}$$

Since $k \geq 1$, by using inequalities $\sinh(k\ell) \leq \cosh(k\ell)$, $\frac{1}{2}e^{k\ell} \leq \cosh(k\ell) \leq e^{k\ell}$, we get

$$\frac{\mathscr{V}_\alpha(k,y)}{k^r} \leq \frac{\mathscr{W}_\alpha(k,y))}{k^r} \leq \frac{2}{\alpha k^r + 2k^r e^{-ky}} \leq \frac{2}{\alpha k^r + 2e^{-k\ell}} = 2\mathscr{R}_r(k). \tag{15}$$

Using the same method as in Section 2, the solution of the regularized problem $\left(\mathscr{P}_\alpha^\delta\right)$ is given by

$$\begin{aligned} v_\alpha^\delta(x,y) \quad = \quad & \sum_{k=1}^{\infty}\left\{\frac{\cosh(ky)}{\cosh_\alpha(k)}c_k(f^\delta) + \frac{\sinh(ky)+\alpha\sinh(k(y-\ell))}{k\cosh_\alpha(k)}c_k(g^\delta) + \right. \\ & \left.\left(\frac{\sinh(k\ell)\left[\alpha\sinh(k(y-\ell))+\sinh(ky)\right]}{2k^2\cosh_\alpha^2(k)} + \frac{y\sinh(k(y-\ell))}{2k\cosh_\alpha(k)}\right)c_k(h^\delta)\right\}\psi_k(x) \end{aligned} \tag{16}$$

where $\cosh_\alpha(k)$ is given by (13).
We are now ready to show that the regularized solution v_α^δ converges to the exact solution v as α goes to zero.

Lemma 2 *Suppose that v_α is the solution of $(\mathscr{P}_\alpha)$ with exact data $E_{data} \in \left(L^2(0,\pi)\right)^3$, and v_α^δ is the solution of problem $(\mathscr{P}_\alpha^\delta)$, with noisy data $E_{data}^\delta \in \left(L^2(0,\pi)\right)^3$. Then the estimate*

$$\left\|v_\alpha^\delta(.,y) - v_\alpha(.,y)\right\|_{L^2(0,\pi)}^2 \quad \leq \quad 8\delta^2\kappa_1(\alpha) + \left(\frac{\delta^2}{\alpha^2}\right)\kappa_2(\alpha) + \left(\frac{\delta^2}{\alpha^4}\right)\kappa_3(\alpha), \tag{17}$$

holds, where $\kappa_1(\alpha) = \dfrac{\ell^2}{\ln^2\left(\frac{2\ell}{\alpha}\right)}$, $\quad \kappa_2 = 1 + 8\kappa_1(\alpha) + 3\kappa_1^2(\alpha)$, *and* $\kappa_3 = 12\kappa_1^2(\alpha)$. □

PROOF Using the Cauchy-Schwartz inequality $\left(\sum_{i=1}^{3} x_i\right)^2 \leq 3\sum_{i=1}^{3} x_i^2$, we can write that:

$$\left\|v_\alpha^\delta(.,y) - v_\alpha(.,y)\right\|_{L^2(0,\pi)}^2 \quad \leq \quad 3\left(\mathbb{T}_f + \mathbb{T}_g + \mathbb{T}_h\right). \tag{18}$$

From (14), one has

$$\mathbb{T}_f \leq \frac{1}{\alpha^2}\left\|\left(f_1^\delta - f_2^\delta\right)\right\|_{L^2(0,\pi)}^2 \leq \frac{\delta^2}{\alpha^2}, \tag{19}$$

It follows from lemma 1 and (15) that

$$\mathbb{T}_g \leq 8\max_{k\geq 1}\left(\mathscr{R}_1^2(k)\right)(1+\alpha^2)\sum_{k=1}^{\infty}\left|c_k\left(g^\delta\right)-c_k(g)\right|^2 \leq \frac{8\ell^2}{\ln^2\left(\frac{2\ell}{\alpha}\right)}\frac{(\alpha^2+1)\delta^2}{\alpha^2}, \tag{20}$$

and

$$\begin{aligned}\mathbb{T}_h &\leq \max_{k\geq 1}\left[3\ell^2\left(\mathscr{R}_1^2(k)\right)+12\alpha^2\left(\mathscr{R}_1^4(k)\right)+12\left(\mathscr{R}_1^4(k)\right)\right]\sum_{k=1}^{\infty}\left|c_k\left(h^\delta\right)-c_k(h)\right|^2\\ &= \frac{3\ell^4\delta^2}{\alpha^2\ln^2\left(\frac{2\ell}{\alpha}\right)}\left(1+\frac{4(1+\alpha^2)}{\alpha^2\ln^2\left(\frac{2\ell}{\alpha}\right)}\right),\end{aligned} \tag{21}$$

■

Combining (19), (20) and (21), we deduce that

$$\left\|w_\alpha^\delta(.,y)-w_\alpha(.,y)\right\|^2_{L^2(0,\pi)} \leq 8\delta^2\kappa_1(\alpha)+\left(\frac{\delta^2}{\alpha^2}\right)\kappa_2(\alpha)+\left(\frac{\delta^2}{\alpha^4}\right)\kappa_3(\alpha), \tag{22}$$

where

$$\kappa_1(\alpha)=\frac{\ell^2}{\ln^2\left(\frac{2\ell}{\alpha}\right)},\quad \kappa_2=1+8\kappa_1(\alpha)+3\kappa_1^2(\alpha), \text{ and } \kappa_3=12\kappa_1^2(\alpha).$$

Remark 1 It is easy to see from (17) that the choice of the regularization parameter α is only related to the noise level δ but not dependent on the a priori bound.

From lemma 2, if we choose $\alpha(\delta)=\delta^\varepsilon$ for some $\varepsilon\in(0,1/2)$, then the error estimate in (17) is of order $\dfrac{\delta^{1-2\varepsilon}}{\ln\left(\frac{2\ell}{\delta^\varepsilon}\right)}$, which tends to zero as $\delta\to 0^+$. □

The conditional stability means that the solution is continuously dependent on the given data under certain additional condition [21]. We assume that the exact data satisfies

$$f\in\mathscr{G}_{1,-2r}\Leftrightarrow \exists C_{1,-2r}>0, \text{ such that } \sum_{k=1}^{\infty}k^{2r}e^{2k\ell}|c_k(f)|^2\leq C_{1,-2r}^2,\ r>0, \tag{23}$$

$$g\in\mathscr{G}_{1,0}\Leftrightarrow \exists C_{1,0}>0, \text{ such that } \sum_{k=1}^{\infty}e^{2k\ell}|c_k(g)|^2\leq C_{1,0}^2, \tag{24}$$

$$h\in\mathscr{G}_{2,0}\Leftrightarrow \exists C_{2,0}>0, \text{ such that } \sum_{k=1}^{\infty}e^{4k\ell}|c_k(g)|^2\leq C_{2,0}^2. \tag{25}$$

Lemma 3 *Let v_α be the solution of the problem $(\mathscr{P}_\alpha)$ corresponding to the exact data. Assume that f,g and h satisfy the conditions (23), (24) and (25) respectively.*

Then,

$$\|v_\alpha(.,y)-v(.,y)\|^2_{L^2(0,\pi)} \leq 12\sigma_r^2(\alpha)C^2_{1,-2r}+C\kappa_1(\alpha)+9\sigma_1^4(\alpha)C^2_{2,0}, \quad (26)$$

where $\sigma_r(\alpha)=\dfrac{\ell_1^r}{\ln^r\left(\frac{\ell_2}{\alpha}\right)}, C=\left(48C^2_{1,0}+9(\ell^2+1)C^2_{2,0}\right), r>0.$ □

PROOF It follows from the Cauchy-Schwartz inequality $\left(\sum_{i=1}^{3}x_i\right)^2 \leq 3\sum_{i=1}^{3}x_i^2$, that

$$\|v_\alpha(.,y)-v(.,y)\|^2_{L^2(0,\pi)} \leq 3\left(\mathbb{L}_f+\mathbb{L}_g+\mathbb{L}_h\right). \quad (27)$$

Thus, by using lemma 1 and (23), coupled with (14), the term $\mathbb{L}_f$ can be estimated as follows

$$\begin{aligned}\mathbb{L}_f &\leq \sum_{k=1}^{\infty}\frac{\alpha^2\cosh^2(ky)\cosh^2(k\ell)}{\cosh^2_\alpha(k)}\frac{k^{2r}}{k^{2r}}|c_k(f)|^2\\ &\leq 4\left(\frac{\ell_1}{\ln\left(\frac{\ell_2}{\alpha}\right)}\right)^{2r}\|f\|^2_{2,-r}\leq 4C^2_{1,-2r}\left(\frac{\ell_1}{\ln\left(\frac{\ell_2}{\alpha}\right)}\right)^{2r}. \end{aligned} \quad (28)$$

Since $g\in\mathscr{G}_{1,0}$, the term $\mathbb{L}_g$ can be estimated as follows

$$\begin{aligned}\mathbb{L}_g &\leq \sum_{k=1}^{\infty}\left(\frac{2\alpha^2\sinh^2(k(y-\ell))}{k^2\cosh^2_\alpha(k)}+\frac{2\alpha^2\sinh^2(ky)\sinh^2(k\ell)}{k^2\cosh^2_\alpha(k)}\right)|c_k(g)|^2\\ &\leq \frac{8\ell^2}{\ln^2\left(\frac{2\ell}{\alpha}\right)}\left(\|g\|^2_{L^2(0,\pi)}+\|g\|^2_{1,0}\right)\leq\frac{16\ell^2}{\ln^2\left(\frac{2\ell}{\alpha}\right)}C^2_{1,0}, \end{aligned} \quad (29)$$

By applying the same techniques with the condition (25), we have

$$\begin{aligned}\mathbb{L}_h &\leq 3(\ell^2+1)\alpha^2\max_{k\geq1}(\mathscr{R}_1^2)\sum_{k=1}^{\infty}e^{2k\ell}|c_k(h)|^2+3\alpha^2\max_{k\geq1}(\mathscr{R}_2^2)\sum_{k=1}^{\infty}e^{4k\ell}|c_k(h)|^2\\ &\leq \left[\frac{3(\ell^2+1)\ell^2}{\ln^2\left(\frac{2\ell}{\alpha}\right)}+3\frac{\ell_1^4}{\ln^4\left(\frac{\ell_2}{\alpha}\right)}\right]C^2_{2,0}, \end{aligned} \quad (30)$$

by regrouping the inequalities (28), (29), (30) and (27) we get the desired estimate (26). ■

Theorem 1 *Let v be the solution of the problem $(\mathscr{P})$ corresponding to the exact data E_{data}. Let v^δ_α be the solution of the perturbed problem $\left(\mathscr{P}^\delta_\alpha\right)$ corresponding to the noisy data E^δ_{data}. Suppose that the regularization parameter α is chosen as $\alpha=\delta^\varepsilon$, $(0<\varepsilon<\frac{1}{2})$, then we have the following estimate*

$$\left\| v_{\alpha}^{\delta}(.,y) - v(.,y) \right\|_{L^2(0,\pi)}^2 \leq \sqrt{12\sigma_r^2 \delta^{\varepsilon} C_{1,-2r}^2 + C\kappa_1\left(\delta^{\varepsilon}\right) + 9\sigma_1^4\left(\delta^{\varepsilon}\right) C_{2,0}^2}$$

$$+ \sqrt{8\delta^2\kappa_1\left(\delta^{\varepsilon}\right) + \delta^{2(1-\varepsilon)}\kappa_2\left(\delta^{\varepsilon}\right) + \delta^{2(1-2\varepsilon)}\kappa_3\left(\delta^{\varepsilon}\right)}, \tag{31}$$

where $\kappa_1\left(\delta^{\varepsilon}\right) = \dfrac{\ell^2}{\ln^2\left(\frac{2\ell}{(\delta^{\varepsilon})}\right)}$, $\quad \kappa_2 = 1 + 8\kappa_1\left(\delta^{\varepsilon}\right) + 3\kappa_1^2\left(\delta^{\varepsilon}\right), \quad \kappa_3 = 12\kappa_1^2\left(\delta^{\varepsilon}\right)$,

$\sigma_r\left(\delta^{\varepsilon}\right) = \dfrac{\ell_1^r}{\ln^r\left(\frac{\ell_2}{(\delta^{\varepsilon})}\right)}$ *and* $\ell_1 = r\ell$, $\ell_2 = 2(\ell)^r/r, C > 0,\ r > 0$. □

PROOF It is clear that

$$\|v_{\alpha}^{\delta}(.,y) - v(.,y)\|_{L^2(0,\pi)} \leq \|v_{\alpha}^{\delta}(.,y) - v_{\alpha}(.,y)\|_{L^2(0,\pi)} + \|v_{\alpha}(.,y) - v(.,y)\|_{L^2(0,\pi)}. \tag{32}$$

Combining lemma 2, lemma 3 and the triangle inequality (32), we obtain the desired estimate. ■

4. Numerical illustrations

Let $\ell = 1$, and $\Omega = \Omega_x \times \Omega_y = [0,\pi] \times [0,1]$. We propose a semi-discrete finite difference scheme with step length $\tau_x = \dfrac{\pi}{N+1}$. For the purpose of numerical illustration, we take the boundary conditions as

$$f(x) = g(x) = h(x) = \sqrt{\frac{2}{\pi}}\sin(x), \tag{33}$$

then, the exact solution $v(x,y)$ with respect to the data (33) is computed by solving the ill-posed problem $(\mathscr{P})$. The analytical form of the solution is given by

$$v(x,y) = \sqrt{\frac{2}{\pi}}\left\{e^y + \frac{1}{2}\sinh(1)\sinh(y) + \frac{y}{2}\sinh(y-1)\right\}\sin(x). \tag{34}$$

Note that in practice, the data $f(x), g(x)$ and $h(x)$ are obtained by measurement, then we generate noisy data by

$$f^{\delta} = g^{\delta} = h^{\delta} = f + \varepsilon randn(size(f)), \tag{35}$$

where $\varepsilon > 0$ indicates the noise level. The bound on the measurement error $E(\delta)$ can be measured according to

$$E(\delta) := \|f^{\delta} - f\|_{l^2} = \left(\frac{1}{N+1} \sum_{i=1}^{N+1} \left(f_i - f_i^{\delta} \right)^2 \right)^{1/2}, \tag{36}$$

For each $y \in \Omega_y$, we evaluate the relative error REr_w by

$$REr r_v = \frac{\|v_{\alpha}^{\delta}(.,y) - v(.,y)\|_{l^2}}{\|v(.,y)\|_{l^2}}. \tag{37}$$

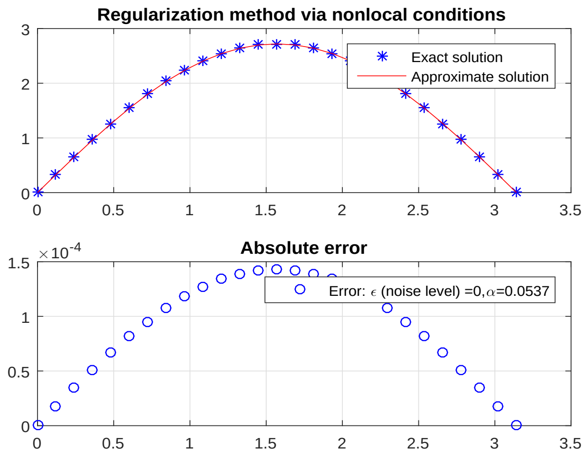

Fig. 1. v and v_{α}^{δ} with exact data $\varepsilon = 0$ for $y = 1$ and the absolute error

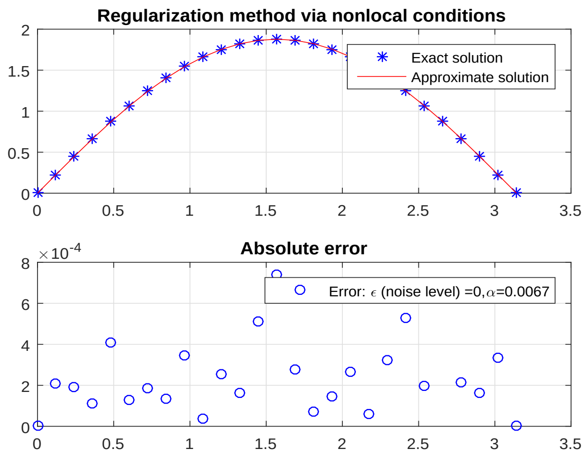

Fig. 2. v and v_{α}^{δ} with exact data $\varepsilon = 0$ for $y = 0.7$ and the absolute error

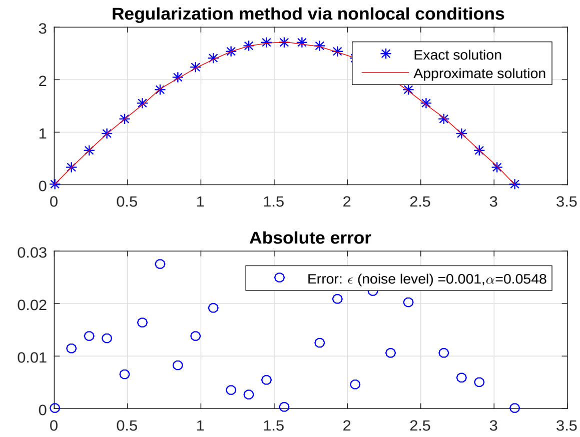

Fig. 3. v and v_α^δ with noisy data $\varepsilon = 0.001$ for $y = 1$ and the absolute error

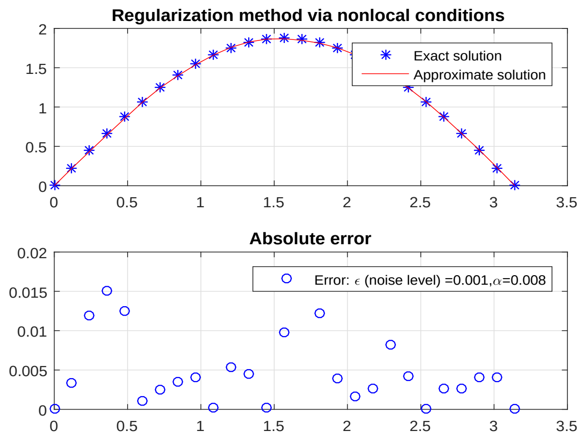

Fig. 4. v and v_α^δ with noisy data $\varepsilon = 0.001$ for $y = 0.7$ and the absolute error

We will pick up 2 specific values of y to illustrate the goodness of fit of the regularized solution: $y = 0.7$ and $y = 1$. We have Figures 1-6, and Tables 1 and 2 to illustrate the result in this situation. According to the a priori choice rule, the numerical results for $v(x,1)$ and $v(x,0.7)$ with exact data are shown in Figures 1 and 2 respectively. The numerical results for $v_\alpha^\delta(x,y)$ are plotted in Figures 3 and 5 at $y = 1$ and in Figures 4 and 6 for $y = 0.7$ with various noisy levels $\varepsilon \in \{0.001, 0.0001\}$. The relative errors between exact and regularized solutions with various noisy levels are shown in Tables 1 and 2. The numerical results presented in all figures show that the regularized solution works very well and yields a very nice approximation to the exact solution. Thanks to all figures, we conclude that the regularized solution is stabilized when y tends to 0. From Tables 2 and 1, one can easily find that smaller values of ε give better regularized solutions.

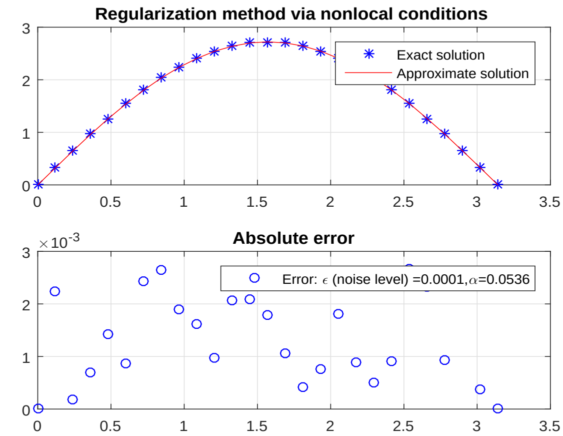

Fig. 5. v and v_{α}^{δ} with noisy data $\varepsilon = 0.0001$ for $y = 1$ and the absolute error

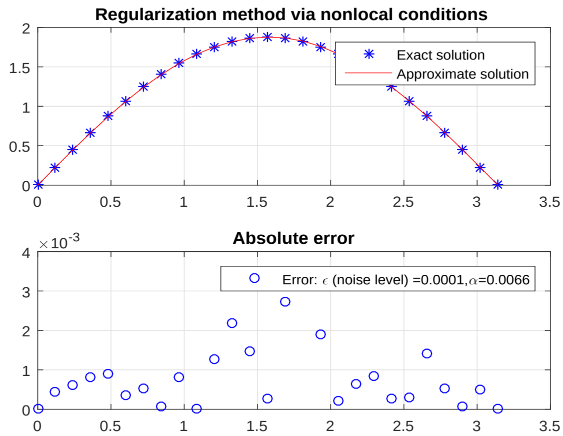

Fig. 6. v and v_{α}^{δ} with noisy data $\varepsilon = 0.0001$ for $y = 0.7$ and the absolute error

5. Conclusions

In this paper, the Cauchy problem associated to the biharmonic equation in a two-dimensional domain has been studied. It has been shown that this problem is ill-posed in the sense of Hadamard. This situation means that the solution does not depend continuously on the given data. In order to solve it, we proposed a regularization method via nonlocal conditions. Convergence and stability estimates, as the noise level tends to zero, are formulated and proved in the setting of a priori parameter choice. The numerical illustration shows that the regularized solution works very well. Our problem is restricted to a rectangle geometry for which the eigenvalues and eigenfunctions of Δ are available. However, if we let an arbitrary domain with

Table 1. The relative errors at $y = 1$ and various noisy levels

Noise level ε	$RErr_w$	Regularization parameter α
0	$5.271 \cdot 10^{-5}$	0.0537
0.01	0.067	0.072
0.001	0.0078	0.0548
0.0001	$8.3449 \cdot 10^{-4}$	0.0563

Table 2. The relative errors at $y = 0.7$ and various noisy levels

Noise level ε	$RErr_w$	Regularization parameter α
0	$2.3458 \cdot 10^{-5}$	0.0067
0.01	0.0351	0.02
0.001	0.0053	0.008
0.0001	$7.2454 \cdot 10^{-4}$	0.0066

a C^2-boundary, the analysis and the regularization method in this paper is not applied. This open problem is a potential future work.

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