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ON THE FRACTIONAL-ORDER DYNAMICS OF A DOUBLE PENDULUM WITH A FORCING CONSTRAINT USING THE NONSINGULAR FRACTIONAL DERIVATIVE APPROACH

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Abstract. In this paper, we presented the fractional-order dynamics of a double pendulum, at a small oscillation, with a non-singular derivative kernel. The equation of motion has been derived from the fractional Lagrangian of the system and the considered fractional Euler-Lagrange equation. The generalized force has also been presented in studying the different cases of force, such as horizontal and vertical forcing. The source term is described by the imposed periodic force, and the memory effect gives an additional damping factor described by the fractional order. The integer and fractional orders of the sample phase diagrams were obtained and presented to visualize the effect of the presented fractional order on the system. Also, since the motion of the system dissipates in the fractional regime, the applied force will drive the system out of equilibrium.

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1. Introduction

In the context of mathematical formulation on calculus, fractional calculus is one of the oldest, yet novel topics, which has attracted many researchers for many decades. Recently, there have been many applications of this within various fields, and, they are growing continuously. It is believed to be powerful in describing non-linear phenomena since one can access a heterogeneity of the system through the given fractional order. To elaborate on the novelty of this fractional calculus topic, we refer to the following papers, and the references therein, to the readers [1–4]. In the past years, studying the dynamics of some physical system, through Lagrange's or Hamilton's equation by generalizing it to the fractional case, intrigued many scientists because of its capability in describing some fractional dimensions. Riewe [5] started these controversial fractional mechanics through the generalized fractional Lagrangian and Hamiltonian of the system for a non-conservative system. Later on, generalized variational problems and Euler-Lagrange equations were established by

Agarwal [6]. With these pieces of evidence of the application of fractional calculus in real-world systems, fractional calculus gives more advantages compared to classical integer calculus because of the convolution of a memory function and the first derivative, in which, more memory effects are accessible [7–12].

In this work, we aim to employ the concept of an exponential memory [13], a non-singular fractional derivative, to the equation of motion of a double pendulum, that is generally absent in the classical model of double pendulum. Additionally, this can be an interesting activity in computational mathematics and physics for undergraduate students. The main features of this paper are presented as follows: To investigate the non-local dynamics of a double pendulum in small oscillation, a fractional derivative operator with the exponential kernel is imposed; A new fractional equation of motion was obtained and the result shows the flexibility of the system due to the imposed operator; Non conservative force conditions were imposed on the system and the explicit analytical solution was also presented; One physical importance observed in the system under the fractional scheme is the additional damping term, which is generally absent in the ideal classical equation of motion for a double pendulum.

This paper is organized as follows: In section 2, we presented some basic definitions and properties of the Caputo-Fabrizio fractional derivative and a brief review on the classical double pendulum with force constraints. We discussed the derivation of the fractional Euler-Lagrange equation in section 3, using the calculus of variation, and presented the fractional description of the system that was discussed in section 4. Simulation of the numerical solution of the fractional equation of motion is presented in section 5 together with the phase diagrams plot. Lastly, the summary and review of the obtained results in this paper were discussed in section 6.

2. Preliminaries

In this section, we provide some preliminaries concerning the utilized fractional derivative, namely, the Caputo-Fabrizio fractional derivative and the classical description of the double from their Lagrangian formulation to the Euler-Lagrange equation.

2.1. Caputo-Fabrizio fractional derivative

This new derivative containing an exponential kernel avoids the singularity of the Caputo fractional derivative. Over the past years, the application of this fractional derivative has been studied in different fields [13–15]. Below, we give the definition and some properties of the utilized Caputo-Fabrizio fractional derivative.

Definition 1 Let f $\in H^1(a,b)$, $b > a$, $\alpha \in (0,1]$, then the Caputo-Fabrizio fractional derivative is defined as

$$ {}^{CF}_{a}D^{\alpha}_{t} f(t) = \frac{C(\alpha)}{1-\alpha} \int_a^t \frac{f(s)}{ds} \exp\Big(-\frac{\alpha}{1-\alpha}(t-s)\Big) ds. \tag{1}$$

where $C(\alpha)$ is the normalization function such that $C(0) = C(1) = 1$. For more details about the property of this derivative, see [13]. □

However, Losada and Nieto [16] showed the explicit form of the equation (1) as

$$ {}_{a}^{CF}D_t^{\alpha} f(t) = \frac{(2-\alpha)C(\alpha)}{2(1-\alpha)} \int_a^t \frac{f(s)}{ds} \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds. \tag{2} $$

where they also write the fractional integral counterpart of (2), given by the definition below.

Definition 2 [16] Let $g(t) = {}^{CF}D_t^{\alpha} f(t)$, then the fractional integral of order α is given by

$$ {}^{CF}I_t^{\alpha} g(t) = f(t) = \frac{2(1-\alpha)}{(2-\alpha)C(\alpha)} g(t) + \frac{2\alpha}{(2-\alpha)C(\alpha)} \int_a^t g(t')dt'. \tag{3} $$ □

Where $C(\alpha)$ is shown to be $C(\alpha) = \dfrac{2}{2-\alpha}$. Note that the classical definition of calculus is recovered if $\alpha \to 1$. For more details about the Caputo-Fabrizio fractional derivative, we refer to papers [13, 16, 17]. We continue the rest of this work with the considerations $C(\alpha) = 1$ and $a = 0$.

2.2. Classical description of double pendulum

In this subsection, we present a brief review of a double pendulum with its Lagrangian, classically. The equation of motion can actually be derived from the classical Euler-Lagrange equation [18],

$$ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i, \tag{4} $$

where q_i and Q_i are the generalized coordinate and the generalized noncoservative force acting on the system. Suppose that the acting force is given by F, then the generalized force is written as

$$ Q_i = \frac{\partial r}{\partial q_i} F. \tag{5} $$

In Figure 1, we have a double pendulum whose masses (m) and lengths (l) are equal and the general coordinates are ϕ_1 and ϕ_2, and the force constraint is acting on the bottom bob with an angle θ with respect to the vertical. Assume that the lengths and masses of the system are equal. So, we can directly deduce the Lagrangian of the system as

$$ L = ml^2\dot{\phi}_1^2 + \frac{ml^2}{2}\dot{\phi}_2^2 + ml^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) + 2mgl\cos(\phi_1) + mgl\cos(\phi_2). \tag{6} $$

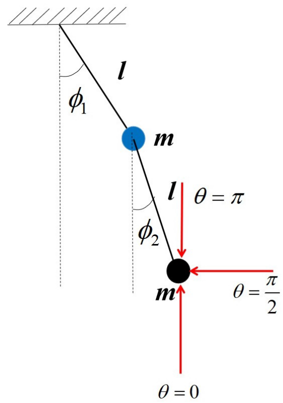

Fig. 1. Forcing conditions applied on the system

In which the force can be written in the form

$$F(t) = \begin{pmatrix} F\cos(\omega t)\sin(\theta) \\ F\cos(\omega t)\cos(\theta) \end{pmatrix}, \tag{7}$$

and the point of application r can be written as

$$r = \begin{pmatrix} l\,(\sin(\phi_1)_+ \sin(\phi_2)) \\ l\,(\cos(\phi_1)_+ \cos(\phi_2)) \end{pmatrix}, \tag{8}$$

then the generalized force is written as

$$Q_i = \begin{pmatrix} Fl\cos(\omega t)\sin(\theta - \phi_1) \\ Fl\cos(\omega t)\sin(\theta - \phi_2) \end{pmatrix}. \tag{9}$$

Substituting equation (5) to (4) yields the classical equation of the motion of pendulum

$$\ddot{\phi}_1 + \frac{1}{2}\cos(\phi_1 - \phi_2)\ddot{\phi}_2 + \frac{1}{2}\dot{\phi}_2^2\sin(\phi_1 - \phi_2) + \frac{g}{l}\sin(\phi_1) = Q_1, \tag{10}$$

$$\ddot{\phi}_2 + \cos(\phi_1 - \phi_2)\ddot{\phi}_1 + \dot{\phi}_1^2\sin(\phi_1 - \phi_2) + \frac{g}{l}\sin(\phi_2) = Q_2. \tag{11}$$

Now, for a small oscillation ($\phi < 15^o$), we can approximately write $\cos(\phi_1 - \phi_2) \approx 1$, $\dot{\phi} \approxeq 0$, and $\sin(\phi) \approxeq \phi$. The equations of motion for both the pendulums are

$$\ddot{\phi}_1 + \frac{1}{2}\ddot{\phi}_2 + \frac{g}{l}\phi_1 = Q_1, \tag{12}$$

$$\ddot{\phi}_2 + \ddot{\phi}_1 + \frac{g}{l}\phi_2 = Q_2. \tag{13}$$

In Figure 1, we show the different orientation of the applied force on the system. For lower vertical forcing, upper vertical forcing, and horizontal forcing, we have the applied force orientation angles $\theta = 0, \pi, \frac{\pi}{2}$, respectively. In the next section, we presented the formulation of the fractional Euler-Lagrange equation from the classical variation principle and the fractionalized Lagrangian.

3. Fractional Euler-Lagrange equation

Imposing the memory effect due to the fractional derivative operator in (2) does not necessarily mean a generalization of the integer counterpart of the derivative. Although the formulation is mathematical, it only gives us an heterogeinity and non-local effect. Nonetheless, this will also give us the response of the system due to the imposed memory effect. Non-local dynamics of the double pendulum can be derived using the fractional derivative operator (2) in the variation principle. Consider a Lagrangian function $L = L(x,\ {}^{CF}D_t^{\alpha} x, t)$ where $x(t)$ is a smooth function and L is considered to have a fractionalize function of $x(t)$. Now the action integral I should also contain a fractional term of the form

$$I = \int_{t_a}^{t_b} L(x,\ {}^{CF}D_t^{\alpha} x, t)dt. \tag{14}$$

The integral should be extremum at $t = t_a$ and t_b. Using the variational principle, we parametrize the function $x(t)$ resulting to a family of curves given by the expression

$$x(\beta, t) = x(0, t) + \beta x(t), \tag{15}$$

such that $\beta(a) = \beta(b) = 0$, and $x(0, t) = x(t)$. Note that the fractional derivative operator is a linear operator, so we have

$${}^{CF}D_t^{\alpha} x(\beta, t) = {}^{CF}D_t^{\alpha}(x(0, t)) + \beta\, {}^{CF}D_t^{\alpha} x(t), \tag{16}$$

substituting (16) to (14) to have

$$I = \int_{t_a}^{t_b} L(x(t),\ {}^{CF}D_t^{\alpha}(x(0, t)) + \beta\, {}^{CF}D_t^{\alpha} x(t), t)dt. \tag{17}$$

Requiring I to extremum at the points t_a and t_b, then we have

$$\frac{dI(\beta)}{d\beta} = \int_{t_a}^{t_b} \frac{d}{d\beta}(L(x(t),\ {}^{CF}D_t^{\alpha}(x(0, t)) + \beta\, {}^{CF}D_t^{\alpha} x(t), t))dt = 0. \tag{18}$$

With the use of a full differential of L and taking the generalized coordinate $x \to q_i$, we can reduce this equation to

$$\int_{t_a}^{t_b} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{{}^{CF}D_t^{\alpha} q_i}\right)\right] \frac{\partial q_i}{\partial \beta} dt = 0. \tag{19}$$

We leave the missing steps to the readers, so we have the fractional Euler-Lagrange equation of the form

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{{}^{CF}D_t^{\alpha} q_i}\right) = 0, \quad x \to q_i. \tag{20}$$

For a non-holonomic system, and using the equation of constraints h_i, we can use the fact

$$\sum_i \sigma_i h_i = 0, \tag{21}$$

where σ_i are the Lagrange's undetermined multipliers and rewrite the fractional Lagrangian as

$$L' = L(q_i, {}^{CF}D_t^{\alpha} q_i, t) + \sum_i \sigma_i h_i. \tag{22}$$

Then the resulting fractional Euler-Lagrange equation with generalized coordinates q_i is given by

$$\frac{d}{dt}\left(\frac{\partial L}{{}^{CF}D_t^{\alpha} q_i}\right) - \frac{\partial L}{\partial q_i} = Q_i. \tag{23}$$

where Q_i's are the known generalized force.

4. Fractional description of the system

Let us now turn our attention to the fractionalized Lagrangian for the double pendulum, so equation (6) can be transformed into its fractional counterpart case as

$$L = ml^2\left({}^{CF}D_t^{\alpha}\phi_1\right)^2 + \frac{ml^2}{2}\left({}^{CF}D_t^{\alpha}\phi_2\right)^2 + ml^2\left({}^{CF}D_t^{\alpha}\phi_1\right)\left({}^{CF}D_t^{\alpha}\phi_2\right) - mgl\phi_1^2 - \frac{mgl}{2}\phi_2^2. \tag{24}$$

In this case, we can apply the introduced fractional Euler-Lagrange equation shown in (23), and we can derive the following equations:

$$\frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_1(t)\right) + \frac{1}{2}\frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_2(t)\right) + \frac{g}{l}\phi_1(t) = Q_i, \tag{25}$$

$$\frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_1(t)\right) + \frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_2(t)\right) + \frac{g}{l}\phi_2(t) = Q_i. \tag{26}$$

Where we consider the small oscillation to also have the condition $\dot{\phi} \approxeq 0$. Equations (25)-(26) are reduced to a classical counterpart when $\alpha \to 1$. The fractionalization of the equation of motion does not necessarily mean a generalized one. The importance of the applied fractional derivative operator can be observed in the response of the system that will be discussed in the subsequent discussions. We can actually solve the equations (25)-(26) numerically or analytically. However, no exact form can be obtained when we solve for the analytical solution but an explicit form. In doing so, we shall linearize the system of equations (25)-(26)

$$\begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_1(t)\right) \\ \frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_2(t)\right) \end{pmatrix} + \frac{g}{l}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}.$$

Then the linearized system of equations can be obtained directly as

$$\begin{pmatrix} \frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_1(t)\right) \\ \frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_2(t)\right) \end{pmatrix} = 2\begin{pmatrix} 1 & -1/2 \\ -1 & 1 \end{pmatrix}\left(-\omega_o^2\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}\right),$$

where $\omega_o = \sqrt{\frac{g}{l}}$. Finally, the fractional equation of motion with a forcing constraint is written as

$$\frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_1(t)\right) = -2\omega_o^2\left(\phi_1(t) - \frac{1}{2}\phi_2(t)\right) + 2\left(Q_1 - \frac{1}{2}Q_2\right), \tag{27}$$

$$\frac{d}{dt}\left({}^{CF}D_t^{\alpha}\phi_2(t)\right) = -2\omega_o^2\left(-\phi_1(t) + \phi_2(t)\right) + 2\left(-Q_1 + Q_2\right). \tag{28}$$

We can then develop a numerical method based on the Adams-Bashforth method for Caputo-Fabrizio fractional derivative operator [19]. Firstly, we consider the reformulations ${}^{CF}D_t^{\alpha}\phi_1(t) = \psi_1(t)$ and ${}^{CF}D_t^{\alpha}\phi_2(t) = \psi_2(t)$, so that we can obtain a system of differential equations

$$\begin{cases} {}^{CF}D_t^{\alpha}\phi_1(t) &=& \psi_1(t), \\ \dot{\psi}_1(t) &=& -2\omega_o^2\left(\phi_1(t) - \frac{1}{2}\phi_2(t)\right) + 2\left(Q_1 - \frac{1}{2}Q_2\right), \\ {}^{CF}D_t^{\alpha}\phi_2(t) &=& \psi_2(t), \\ \dot{\psi}_2(t) &=& -2\omega_o^2\left(-\phi_1(t) + \phi_2(t)\right) + 2\left(-Q_1 + Q_2\right). \end{cases}$$

Note that we can get the further simplified form of the system of differential equations above by implementing the fractional integral defined in definition (2) to get the explicit form

$$\begin{cases} \phi_1(t) &=& \phi_1(0) + (1-\alpha)\psi_1(t) + \alpha\int_0^t \psi_1(t')dt', \\ \psi_1(t) &=& -2\omega_o^2\int_0^t \left(\phi_1(t') - \frac{1}{2}\phi_2(t')\right)dt' + 2\int_0^t\left(Q_1 - \frac{1}{2}Q_2\right)dt', \\ \phi_2(t) &=& \phi_2(0) + (1-\alpha)\psi_2(t) + \alpha\int_0^t \psi_2(t')dt', \\ \dot{\psi}_2(t) &=& -2\omega_o^2\int_0^t \left(-\phi_1(t') + \phi_2(t')\right)dt' + \int_0^t 2\left(-Q_1 + Q_2\right)dt'. \end{cases}$$

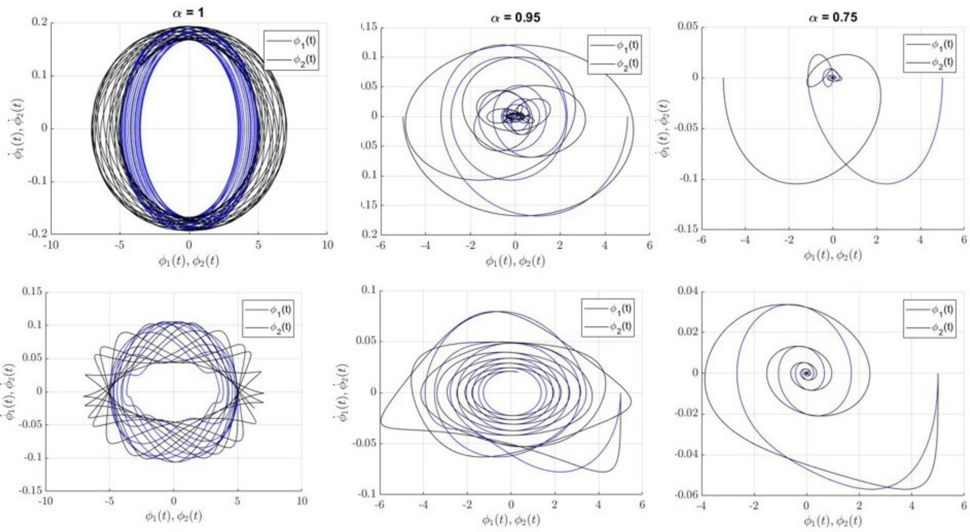

Fig. 2. The phase diagrams for the free oscillation of the double pendulum with a different set initial condition, Upper: Set 2; Lower: Set 3

Now, consider a uniform time mesh $[t_a, t_b]$ with the increment h and time nodes are $t_n, n = 0, 1, 2, ..., N$ where N is the grid size. Imposing the method in [19] to discretize the fractional integral, we have the final form of numerical approximation

$$\begin{cases} \phi_{1,n}(t) &= \phi_{1,n-1} + \left((1-\alpha) + \frac{3}{2}h\alpha\right)\psi_{1,n-1}(t) + \left((1-\alpha) + \frac{1}{2}h\alpha\right)\psi_{1,n-2}(t), \\ \psi_{1,n-1} &= \psi_{1,n-2} + hM_1(\phi_{1,n-2}, \phi_{2,n-2}, t_{n-2}), \\ \phi_{2,n}(t) &= \phi_{2,n-1} + \left((1-\alpha) + \frac{3}{2}h\alpha\right)\psi_{2,n-1}(t) + \left((1-\alpha) + \frac{1}{2}h\alpha\right)\psi_{2,n-2}(t), \\ \psi_{2,n-1} &= \psi_{2,n-2} + hM_2(\phi_{2,n-2}, \phi_{2,n-2}, t_{n-2}), \end{cases}$$

where

$$M_1 = -2\omega_o^2\left(\phi_1(t) - \frac{1}{2}\phi_2(t)\right) + 2\left(Q_1 - \frac{1}{2}Q_2\right),$$
$$M_2 = -2\omega_o^2\left(-\phi_1(t) + \phi_2(t)\right) + 2\left(-Q_1 + Q_2\right). \quad (29)$$

Imposing the generalized force constraint shown in (9), we can then do the numerical simulation as discussed in the next section.

5. Simulation results

We take the following values parameter for the simulations: $g = 9.81$, $m = 1$ kg, $l = 1$ m, $N = 500$, $h = 0.01$. The set of initial values are considered to show the

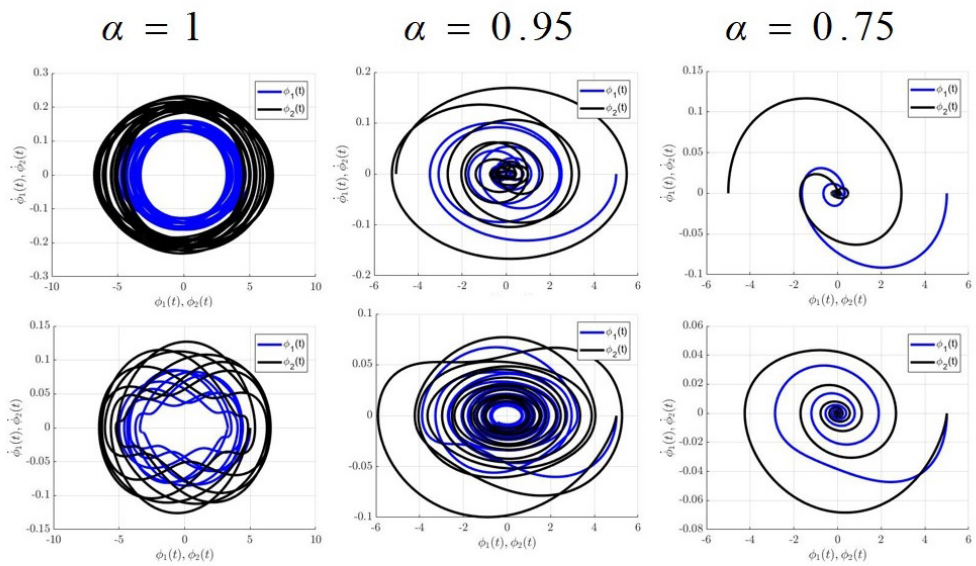

Fig. 3. The phase diagrams for the forcing at the top ($\theta = \pi$) of the bottom bob with a different set initial condition, Upper: Set 2; Lower: Set 3

dependence of the motion on the initial values given by:

	$\phi_1(0)$	$\phi_2(0)$	$\phi'_1(0)$	$\phi'_2(0)$
Set 1: At Rest	0	0	0	0
Set 2: Symmetric	5^o	-5^o	0	0
Set 3: Anti-Symmetric	5^o	5^o	0	0

We have provided the plot of the phase diagrams of the system considering the given set of the initial condition. These (Figs. 2-5) indicate an important effect of the imposed fractional-order as it gives a more flexible solution and dynamics. We can directly notice the dissipation of the system as α decreases. It would physically signify that the fractional-order α acts as a damping factor, or a friction coefficient of the system. As α decreases, the damping factor increases, thus, the system is driven by the force constraint out of the equilibrium. In the subsequent figures, we have considered a different orientation of forcing and interesting phase diagrams are shown. As shown in the simulations, the system of double pendulum acquires different trajectories for different values of the fractional order α, which shows an interesting feature on the system's dynamics. A possible extension of this work could be carried out by considering a conservative force to have more insights into the system.

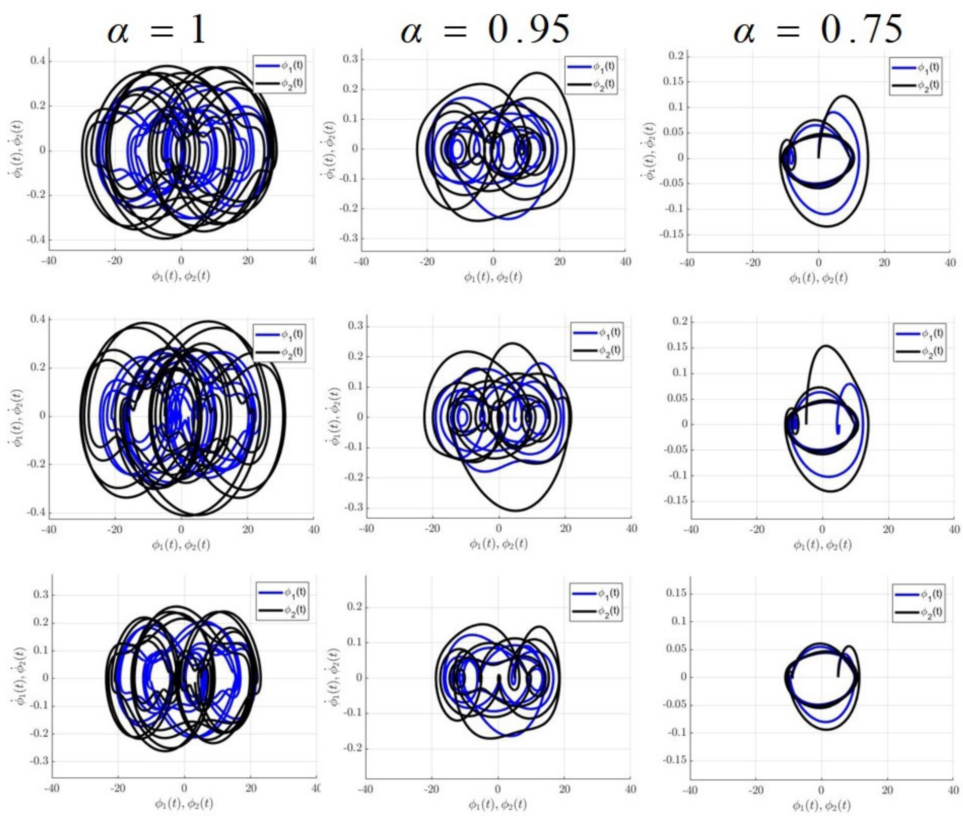

Fig. 4. The phase diagrams of the double pendulum with a horizontal forcing $\theta = \frac{\pi}{2}$ different set initial condition, Upper: Set 2; Lower: Set 3

6. Discussion

After implementing a computer simulation for the fractional equation of motion, we have extracted the phase diagrams corresponding to a different set of the initial condition and forcing type. In Figure 2, the phase diagram of free oscillation was illustrated. For $\alpha = 1$, we have extracted the phase diagram corresponding to a classical free oscillation of a double pendulum, and the system is conserved. However, for $\alpha < 1$, the phase diagram exhibits a damped system since the energy dissipates. Additionally, we can also observe that different dynamics are observed for different types of the initial condition, which the result is not surprising because this is the physical property of a classical free double pendulum. However, if we look at the system with forcing constraint, as we did in Figure 4, the different trajectory was observed for a different set initial conditions. Interestingly, with the inclusion of the force constraint, the system does not dissipate entirely. For a the decrease in α, the damping effect, due to the imposed memory of fractional derivative, increases. In fact, the external force will continue to drive the system out of equilibrium for

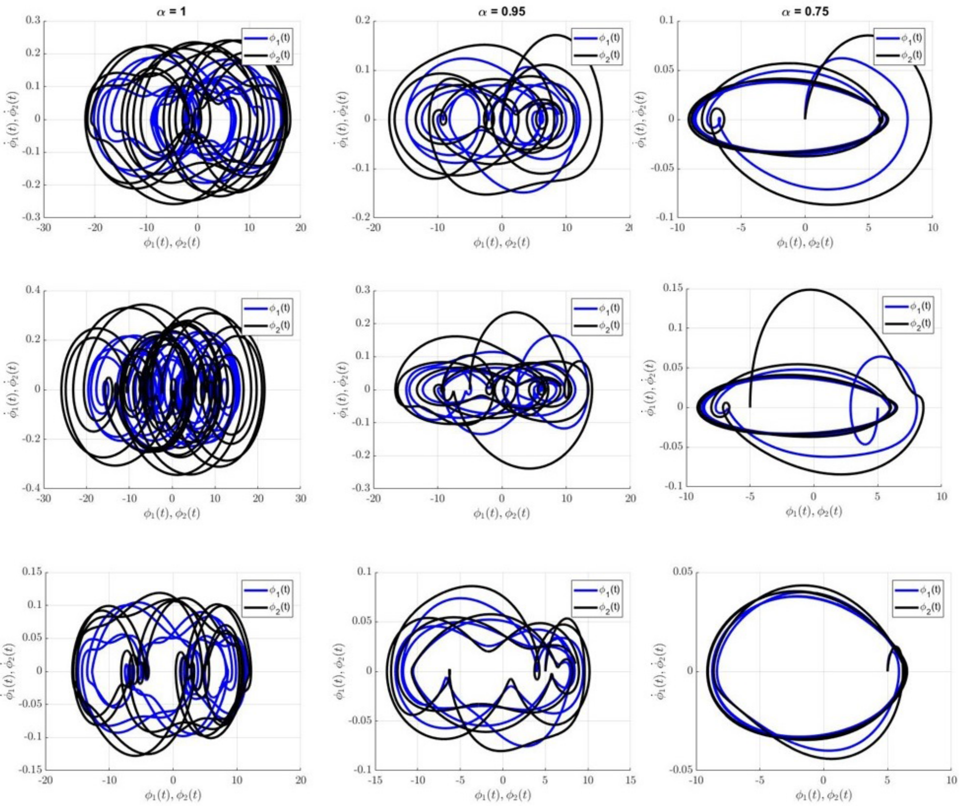

Fig. 5. The phase diagrams of the double pendulum with axial forcing $\theta = 0$ with a different set initial condition, Upper: Set 2; Lower: Set 3

a long period of time. This kind of aspect is important in the controlled vibration. Classically, if the system has an external driving force, the system's oscillation will tend to increase rapidly causing more vibration. With the help of the fractional-order, this kind of problem can be avoided and the external force will drive the system steadily.

7. Conclusion

We have investigated the fractional-order equation of motion of the double pendulum in the fractional order scheme, numerical solutions were obtained and simulation was implemented. It has been shown, from the simulations, that the fractional-order derivative imposed on the double pendulum equation of motion can result in a damping-like effect, which can be a direct response of the system due to the memory effect. However, this observed damping-like effect of the fractional-order does not mean physical damping on the system, unless more experimental works can be provided to support the simulation done in this study. Lastly, the fractional-order regime gives more insights and possible physical relevance than it does in the integer-order case.

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