ON A RECURRENCE FOR PERMANENTS OF A SEQUENCE OF 3-TRIDIAGONAL MATRICES

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Abstract. This is a corrigendum of the paper: Kılıç, A. Z. & Düz, M. (2017). Relationships between the permanents of a certain type of k-tridiagonal symmetric Toeplitz and the Chebyshev polynomials. Journal of Applied Mathematics and Computational Mechanics, 16, 75-86. We will show that Remark 9, on page 84, does not hold, what is the consequence of the incorrect proof, which authors formulated there.

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1. Introduction

The so-called k-tridiagonal matrices (this name was introduced by El-Mikkawy and Sogabe [1]) were first studied by Egerváry and Szász in [2]. Perhaps the most important non-trivial case is due to Losonczi [3]. A very recent and important survey in this topic can be found in da Fonseca and Kowalenko [4].

The k-tridiagonal matrices $T_{n}^{(k)}(D_{-k}, D_{0}, D_{k})$ are defined by the following way

$$
\begin{pmatrix}
 d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 \\
 0 & d_2 & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{n-k} \\
 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
 b_{k+1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 0 & \cdots & 0 & b_n & 0 & \cdots & 0 & d_n
\end{pmatrix}_{n \times n}
$$
where sequences \(\{d_j\}_{j=1}^n\), \(\{a_j\}_{j=1}^{n-k}\) and \(\{b_j\}_{j=k+1}^n\) create the main diagonal \(D_0\), the \(k\)-th superdiagonal \(D_k\) and the \(k\)-th subdiagonal \(D_{-k}\), respectively. Thus, for the general \(k\)-tridiagonal matrix we use notation \(T_n^{(k)}(D_{-k}, D_0, D_k)\) or directly

\[ T_n^{(k)}(\{b_j\}_{j=k+1}^n, \{d_j\}_{j=1}^n, \{a_j\}_{j=1}^{n-k}) \]

but for the \(k\)-tridiagonal Toeplitz matrix we can write shortly \(T_n^{(k)}(b, d, a)\), since for diagonals of matrix (1) hold

\[ \{d_j = d\}_{j=1}^n, \{a_j = a\}_{j=1}^{n-k}, \text{ and } \{b_j = b\}_{j=k+1}^n \]

Küçük, Düz [5] studied, recursive relations between the Chebyshev polynomials of the second kind (for more information, see [6]), which can be defined for \(n > 2\) by the recurrence relation

\[ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \]

with initial values \(U_0(x) = 1\) and \(U_1(x) = 2x\), and the permanents (the definition and many properties of permanents you can find in [7]) of a special type of matrix (1), namely \(k\)-tridiagonal symmetric Toeplitz matrix \(T_n^{(k)}(i, 2x, i)\), where \(i\) is the imaginary unit, i. e., the matrix with entries

\[ t_{jm}^{(k)} = \begin{cases} 2x, & j = m; \\ i, & j = m \pm k; \\ 0, & \text{otherwise} \end{cases} \]

where \(1 \leq j, m \leq n\).

To prove [5, Conjecture 8] first da Fonseca in [8] showed that the permanent of the matrix \(T_n^{(k)}(i, 2x, i)\) is equal to the permanent of the matrix \(T_n^{(k)}(-1, 2x, 1)\), with respect to the fact, that the permanent of a square matrix equals the sum of the weights of all cycle-covers of its underlying directed graph. Then, he used a result on convertible matrices from his paper [10] (some generalizations can be found in [11]) to show that the permanent of matrix \(T_n^{(k)}(-1, 2x, 1)\) is equal to the determinant of the matrix \(T_n^{(k)}(1, 2x, 1)\). Thus, he derived that

\[ \text{per} T_n^{(k)}(i, 2x, i) = \text{det} T_n^{(k)}(1, 2x, 1) \quad (2) \]

Borowska et al. [12–14] dealt with determinants of some pentagonal and heptadiagonal symmetric Toeplitz matrices. Inter alia, they paid attention to the determinant of the following heptadiagonal matrix

\[ T_n^{(k)}(b, d, a) \]

\(^1\)Here we use the notation for the numbering diagonals, which can be found, e. g., in [9].
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\[ \mathbf{A}_n = \begin{pmatrix} a & b & c & d \\ b & a & b & c & d \\ c & b & a & b & c \\ d & c & b & a & b & c & d \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ d & c & b & a & b & c & d \\ d & c & b & a & b & c \\ \end{pmatrix} \]

To find a recurrence relation for determinants of matrix \( \mathbf{A}_n \) they introduced the following two auxiliary heptadiagonal matrices

\[ \mathbf{\Lambda}_n = \begin{pmatrix} a & b & c & d \\ b & a & b & c & d \\ c & b & a & b & c \\ d & c & b & a & b & c & d \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ d & c & b & a & b & c \\ d & c & b & a & b & 0 \\ \end{pmatrix} \]

\[ \mathbf{\tilde{\Lambda}}_n = \begin{pmatrix} a & b & c & d \\ b & a & b & c & d \\ c & b & a & b & c \\ d & c & b & a & b & c & d \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ d & c & b & a & b & c & 0 \\ d & c & b & a & b & d \\ \end{pmatrix} \]

They denoted determinants of matrices \( \mathbf{A}_n, \mathbf{\Lambda}_n, \) and \( \mathbf{\tilde{\Lambda}}_n \) by \( W_n, \mathbf{\tilde{W}}_n \), and \( \mathbf{\hat{W}}_n \), respectively, and derived the following system of linear recurrence relations (see formulae (4) and (5) in [14], where all the needed initial conditions can be found too)
\[ W_{n+7} = a W_{n+6} + bd(bd - 2c^2)W_{n+3} + d^2(2c^3 - 4bcd + b^2c + ad^2)W_{n+2} + d^3(2c^2d + b^2d - bc^2 - a^2)W_{n+1} - bcd^5 W_n - b \bar{W}_{n+6} + bc \bar{W}_{n+5} + d(2ac - b^2)\bar{W}_{n+4} + bd^2(2c - a)\bar{W}_{n+3} + d^3(2bd - b^2 - c^2)\bar{W}_{n+2} + cd^4(b - 2d)\bar{W}_{n+1} + bd^6 \bar{W}_n - c^2 \bar{W}_{n+5} + d(bc - ad)\bar{W}_{n+4}, \]

\[ \bar{W}_{n+6} = b W_{n+5} - bc(b - 2d)W_{n+2} + d^3(c^2 - bd)W_{n+1} + cd^5 W_n - c \bar{W}_{n+5} + bd \bar{W}_{n+4} + ad^2 \bar{W}_{n+3} + bd^3 \bar{W}_{n+2} - cd^4 \bar{W}_{n+1} - d^6 \bar{W}_n - cd \bar{W}_{n+4}, \]

\[ \bar{W}_{n+2} = a W_{n+1} - c^2 W_n + 2cd \bar{W}_n - d^2 \bar{W}_n \]

2. Main result

Küçük, Düz [5] formulated the following proposition (we have made a small technical textual modification, that does not change their assertion, to avoid copying the whole text above this proposition)

**Remark 1**

\[ \text{per} \ T_n^{(3)}(i, 2x, i), \ \text{per} \ T_n^{(4)}(i, 2x, i), \ \text{per} \ T_n^{(5)}(i, 2x, i), \ldots \]

cannot be written in terms of themselves, thus as a self-recurrence for every of these permanents individually. □

Küçük, Düz formulated the proof of this Remark 1 for the case \( \text{per} \ T_n^{(3)}(i, 2x, i) \), but the idea of this proof is incorrect, what we show by proving that there is a self-recurrence for \( \text{per} \ T_n^{(3)}(i, 2x, i) \).

For the simplification of notation, we will use for permanent of matrix \( T_n^{(3)}(i, 2x, i) \) the following denotation

\[ p_n := \text{per} \ T_n^{(3)}(i, 2x, i) \]

where \( n \) is a positive integer.

**Theorem 1** Let \( n \) be any positive integer. The sequence \( \{p_n\} \), defined by (6), satisfies the following recurrence relation for \( n > 8 \)

\[ p_n = 2x p_{n-1} - p_{n-2} + 2x p_{n-3} - 4x^2 p_{n-4} + 2x p_{n-5} - p_{n-6} + 2x p_{n-7} - p_{n-8} \]

with the initial values...
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\[ p_1 = 2x, \quad p_2 = 4x^2, \quad p_3 = 8x^3, \]
\[ p_4 = 4x^2(4x^2 - 1), \quad p_5 = 2x(4x^2 - 1)^2, \]
\[ p_6 = (4x^2 - 1)^3, \quad p_7 = 4x(2x^2 - 1)(4x^2 - 1)^2, \]
\[ p_8 = (4x)^2(2x^2 - 1)^2(4x^2 - 1) \]

**Proof** Combining identities (2) and (6) we get \( p_n = \det T_n^{(3)}(1, 2x, 1) \), but this determinant is a special case of the determinant of the heptadiagonal matrix \( A_n \) in (3), when we set \( a = 2x, b = c = 0, \) and \( d = 1 \). Similarly, we denote determinants of matrices \( \overline{A}_n \) and \( \hat{A}_n \) by \( \overline{p}_n \) and \( \hat{p}_n \), respectively. Then, from (4) we get the following system of three homogeneous linear recurrences for sequences \( \{p_n\}, \{\overline{p}_n\} \) and \( \{\hat{p}_n\} \)

\[
\begin{align*}
p_{n+6} &= 2xp_{n+5} + 2xp_{n+1} - p_n - 2x\hat{p}_{n+3}, \\
\overline{p}_{n+6} &= 2x\overline{p}_{n+3} - \overline{p}_n, \\
\hat{p}_{n+2} &= 2xp_{n+1} - \hat{p}_n
\end{align*}
\]

Since we are only interested in the sequence \( \{p_n\} \), we can omit the second recurrence from the previous system to take the following system of two linear recurrences for sequences \( \{p_n\} \) and \( \{\hat{p}_n\} \)

\[
\begin{align*}
p_{n+6} &= 2xp_{n+5} + 2xp_{n+1} - p_n - 2x\hat{p}_{n+3}, \\
\hat{p}_{n+2} &= 2xp_{n+1} - \hat{p}_n
\end{align*}
\]

which can be easily reduced by substitution method to the self-recurrence (7) of the sequence \( \{p_n\} \). Initial conditions (8) for \( p_i, 1 \leq i \leq 7 \), we easily get as special cases of (5) in [14] and the initial condition for \( p_8 \) we can compute from (4) in [14]. Thus, the proof is complete.

3. Conclusions

In this article, our main purpose was to show that the statement in [5, Remark 9] is incorrect. For this purpose, we have found the self-recurrence for the sequence of permanents of the 3-tridiagonal Toeplitz matrix \( T_n^{(3)}(i, 2x, i) \). Our derivation was based on two substantial previous results. First, we used da Fonseca [8], in which the author showed that the permanent of matrix \( T_n^{(k)}(i, 2x, i) \), studied by Küçük and Düz [5], is equal to the determinant of the matrix \( T_n^{(k)}(1, 2x, 1) \). Subsequently, we used Borowska and Łacińska [14], in which authors found the recurrence system for calculating determinants of the heptadiagonal Toeplitz matrices.
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References


