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ON VALUES OF THE PSI FUNCTION

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Abstract. In the present paper, values of the psi function for many arguments connected with the golden ratio and Fibonacci numbers are determined or given in alternative form. Moreover, some integral representation of the psi function is found. This is a potential calculation base of values of the psi function for powers of argument. We also note that this integral representation gives better numerical estimation of values of the psi function than the respective Legendre's integral formula.

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1. Introduction

The present paper concerns the values of the psi function in arguments connected with the golden ratio (and in consequence with Fibonacci and Lucas numbers) as well as with powers of argument in general form. The main reason for our interest in this area is a lack of results of this type in the literature [1-9]). Of course, we would like to fill this gap. Fundamental results obtained here (Theorems 2 and 3) substantially enrich (and complete) the state of knowledge of the considered topic. Moreover, these results also show that there are still some new areas of applications of the golden ratio, Fibonacci numbers and corresponding recurrence equations (for example as an another and simultaneously new area of such application - see the contents of our recent published paper [10]). We emphasize that experience acquired while writing the book [7] - particularly by the last author, were important in creating this paper. They also had a direct impact on the content of research taken here.

2. Technical introduction

The technical realization of our project requires a number of preparations and it forced us to derive many additional relations. Let us start with recalling basic notions and facts related to the psi function.

The psi function is defined to be the logarithmic derivative of the gamma function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \qquad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

It constitutes one of the most important special functions, which occur today innearly all fundamental analytic areas of mathematics. Let us recall some basic identities for the psi function which will be applied in the sequel. The majority of them can be found in [7] (see also [1, 2]). Namely, we have

$$\psi(1 - z) = \psi(z) + \pi \cot \pi z,$$

$$\psi(z + n) = \psi(z) + \sum_{k=0}^{n-1} \frac{1}{z + k},$$

$$\psi(z + 1) = \psi(z) + \frac{1}{z}, \tag{1}$$

$$\psi(z + 1) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z + n}\right), \tag{2}$$

where γ is the Euler constant (see [7]). Moreover, φ will denote here the golden ratio (so $\varphi = \frac{1+\sqrt{5}}{2}$).

The following lemma is crucial for our considerations, especially in the proof of Theorem 3. Let us emphasize that this technical result has a wide range of applications (see additionally [11] or [12]), and as far as identities a) - d) are known, the result from point e) seems to be original. Although identities a) - d) are not new, they are very often presented in different forms. So, for the reader's convenience, we have decided to add the proof of the point c) - the most intriguing case among the presented ones.

Lemma 1. *The following identities hold true:*

a)

$$\lfloor F_n \sqrt{5} \rfloor = L_n - \frac{(-1)^n + 1}{2}, \qquad n = 2, 3, 4, \dots,$$

b)

$$\lfloor F_n\varphi\rfloor = F_{n+1} - \frac{(-1)^n+1}{2}, \qquad n = 1,2,3,\dots, \tag{3}$$

c)

$$\lfloor L_n\sqrt{5}\rfloor = 2L_n + L_{n-3} + \frac{(-1)^n-1}{2}, \qquad n = 4,5,6,\dots,$$

d)

$$\lfloor L_n\varphi\rfloor = L_{n+1} + \frac{(-1)^n-1}{2}, \qquad n = 1,2,3,\dots,$$

e) Let $p \in \mathbb{Z}[x]$ *and suppose that*

$$p(x) = x^n + a_1x^{n-1} + \cdots + a_{n-k}x^k,$$

where $k \geq 1$*. Then we get*

$$\lfloor p(\varphi^r)\rfloor = sp(L_r) - \frac{1+(-1)^{kr}\operatorname{sgn}(a_{n-k})}{2},$$

$$\lfloor\sqrt{5}p(\varphi^r)\rfloor = 5sp(F_r) - \frac{1-(-1)^{kr}\operatorname{sgn}(a_{n-k})}{2}, \tag{4}$$

for sufficiently large $r \in \mathbb{N}$*, where*

$$sp(b_m) := b_{mn} + a_1b_{m(n-1)} + \cdots + a_{n-k}b_{mk}, \ m \in \mathbb{N},$$

i.e. $\{sp(b_m)\}_{m=1}^{\infty}$ *is the sequence of convolution type sums connected with polynomial p and a given sequence* $\{b_m\}_{m=1}^{\infty}$.

Proof. a) - d) All proofs can be done by using the Binet's formulae for F_n and L_n, respectively. We only prove identity c). First, observe that

$$\sqrt{5}L_n - 2L_n - L_{n-3} = (2\varphi-3)(\varphi^n+\bar\varphi^n) - \varphi^{n-3} - \bar\varphi^{n-3}$$

$$= (2\varphi^4 - 3\varphi^3 - 1)\varphi^{n-3} - (2\bar\varphi^2 + 3\bar\varphi^3 + 1)\bar\varphi^{n-3},$$

where $\bar\varphi = \frac{1-\sqrt{5}}{2}$. We note that $\delta^2 = \delta + 1$, so

$$\delta^3 = \delta^2 + \delta, \qquad \delta^4 = \delta^2 + 2\delta + 1$$

for $\delta = \varphi$ and $\delta = \bar\varphi$, which implies

$$2\varphi^4 - 3\varphi^3 - 1 = -\varphi^2 + \varphi + 1 = 0,$$

and

$$-2\bar{\varphi}^2 - 3\bar{\varphi}^3 - 1 = -5\bar{\varphi}^2 - 3\bar{\varphi} - 1 = -8\bar{\varphi} - 6 = -10 + 4\sqrt{5} \approx -1.0557281.$$

Then we obtain

$$\sqrt{5}L_n - 2L_n - L_{n-3} = (-1)^n(1.055728 + \varepsilon_n)(0.6180339 + \gamma_n)^{n-3},$$

which for $n \geq 4$ implies c) because $1.055728 \cdot 0.6180339 < 1$. The formula c) does not hold for $n = 0, 1, 2, 3$.

e) We have

$$\begin{aligned}\sqrt{5}p(\varphi^r) &= \sqrt{5}\big(p(\varphi^r) - p(\bar{\varphi}^r)\big) + \sqrt{5}p(\bar{\varphi}^r)\\ &= \sqrt{5}(\varphi^{rn} - \bar{\varphi}^{rn}) + \sqrt{5}a_1\big(\varphi^{r(n-1)} - \bar{\varphi}^{r(n-1)}\big) + \cdots\\ &+\sqrt{5}a_{n-k}\big(\varphi^{rk} - \bar{\varphi}^{rk}\big) + \sqrt{5}a_{n-k}\bar{\varphi}^{rk} + o\big(\bar{\varphi}^{rk}\big)\\ &= 5F_{rn} + 5a_1F_{r(n-1)} + \cdots + 5a_{n-k}F_{rk} + \sqrt{5}a_{n-k}\bar{\varphi}^{rk} + o\big(\bar{\varphi}^{rk}\big)\\ &= 5sp(F_r) + \sqrt{5}a_{n-k}\bar{\varphi}^{rk} + o\big(\bar{\varphi}^{rk}\big),\end{aligned}$$

from which the relation (4) follows directly.

3. Main result for powers of argument

The main goal of this section is to present the integral type formula for the psi function for powers of argument. As it is shown below, our integral formula (5) is better than the classic Legendre's one (see (6)).

Theorem 2. The function psi satisfies the following integral relation:

$$\psi(t) - \psi(t^n) + (n-1)\log t = \int_0^1 \left(\frac{1}{1-x} - \frac{t}{1-x^t}\right)\left(\sum_{k=2}^{n} x^{t^{k-1}}\right)dx, \quad (5)$$

where $n \in \mathbb{N} \setminus \{1\}$ *and* $t > 0,\ t \neq 1$.

Proof. We note that

$$\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{t}{1-x^t}\right) = \lim_{x\to 1}\frac{1 - t + tx - x^t}{(1-x)(1-x^t)} = \frac{1-t}{2}$$

by applying the L'Hospital's rule twice. Let us use the following Ramanujan integral (see [13, 14]):

$$\psi\left(\frac{q}{r}\right) - \psi(p) + \log r = \int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{r x^{q-1}}{1-x^r}\right) dx.$$

Now, put $p = q = t^k$ and $r = t$. Summing up these equalities side by side for $k \in \{2,3,\dots,n\}$, we get

$$\psi(t) - \psi(t^n) + (n-1)\log t = \int_0^1 \left[\frac{1}{1-x}\left(\sum_{k=2}^{n} x^{t^k-1}\right) - \frac{t}{1-x^t}\left(\sum_{k=2}^{n} x^{t^k-1}\right)\right] dx$$
$$= \int_0^1 \left(\frac{1}{1-x} - \frac{t}{1-x^t}\right)\left(\sum_{k=2}^{n} x^{t^k-1}\right) dx,$$

which is the desirable equality (5).

Referring to equation (5) one might ask whether it would be better to use the following Legendre's formula (see [7], section 4.3)

$$\psi(t) = -\gamma + \int_0^1 \frac{x^{t-1}-1}{x-1} dx, \tag{7}$$

to calculate $\psi(t^n)$? It is easy to check that if we replace t by t^n and n is sufficiently large, then the numerical convergence of the integral from (6) is worse than convergence of the integral from (5). The following relations explain it better:

$$\frac{x^{t^n-1}-1}{x-1} \underset{y=1-x}{\overset{x\approx 1}{=}} \frac{(1-y)^{t^n-1}-1}{-y} \overset{t>1}{=} (t^n-1) - \frac{1}{2!}(t^n-1)(t^n-2) + \cdots$$

and

$$\frac{1-t+tx-x^t}{(1-x)(1-x^t)} \underset{y=1-x}{\overset{x\approx 1}{=}} \frac{1-ty-(1-y)^t}{y(1-(1-y)^t)}$$
$$= \frac{-\binom{t}{2}y^2 + \binom{t}{3}y^3 - \cdots}{y^2\left(t - \binom{t}{2}y + \binom{t}{3}y^2 - \cdots\right)} = \frac{-\binom{t}{2} + \binom{t}{3}y - \cdots}{t - \binom{t}{2}y + \binom{t}{3}y^2 - \cdots}.$$

Therefore it is better to apply the formula (5) to find $\psi(t^n)$.

We propose comparing the integral relation (5) with the asymptotic formula from the point f) of Theorem 3.

It is very surprising that for the psi function there is another couple of formulae of type (5) and (6), namely

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n+x}=\log 2-\psi(x)+\psi\left(\frac{x}{2}\right)+\frac{1}{x}, \tag{7}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n(n+x)}=\frac{1}{x}\left(\psi(x)-\psi\left(\frac{x}{2}\right)-\frac{1}{2}\right). \tag{8}$$

The proof of the formula (8) can be found in [15]. Formulae (7) and (8) differ from (5) and (6) by the speed of convergence ofthe respective functional series occurring there. The identity (7) and especially the identity (8) by appropriate iterations of these formulae can be used to calculate the value of $\psi(x)$ for large values of x.

4. The values of the psi function in arguments connected with the golden ratio and Fibonacci numbers

The values of the psi function for many different arguments connected with the golden ratio (especially for powers of φ) as well as Fibonacci and Lucas numbers will be shown in our secondmain theorem.

Theorem 3. *The following identities hold:*

a)

$$\psi(x+\varphi)-\psi\left(x-\frac{1}{\varphi}\right)=\psi(x+\varphi)-\psi(x+1-\varphi)$$
$$=\sum_{k=1}^{\infty}\frac{2\varphi-1}{(x+k)^2-(x+k)-1}=\sum_{\tau=0}^{\infty}\frac{\sqrt{5}}{(x+\tau)^2+(x+\tau)-1};$$

b)

$$\psi(\varphi^n)-\psi(1-\varphi^n)=\begin{cases}-\pi\cot\left(\pi\frac{F_n\sqrt{5}}{2}\right) & \text{if } 3/n\\ \pi\tan\left(\pi\frac{F_n\sqrt{5}}{2}\right) & \text{otherwise;}\end{cases}$$

c)

$$\psi(\lfloor\varphi^n\rfloor)=\begin{cases}-\gamma+H_{L_{n-2}} & \text{if } 2/n,\\ -\gamma+H_{L_{n-1}} & \text{otherwise,}\end{cases}$$

where H_m denotes the m-th harmonic number

$$H_m:=\sum_{k=1}^{m}\frac{1}{k},\qquad m\in\mathbb{N};$$

d)

$$\psi(F_n\sqrt{5}) = \begin{cases} \psi(F_n\sqrt{5} - L_n + 2) + \sum_{k=0}^{L_n-3} \frac{1}{k + F_n\sqrt{5} - L_n + 2} & \text{if } 2|n, \\ \psi(F_n\sqrt{5} - L_n + 1) + \sum_{k=0}^{L_n-2} \frac{1}{k + F_n\sqrt{5} - L_n + 1} & \text{otherwise.} \end{cases}$$

Moreover, if $2|n$ *then*

$$1 < F_n\sqrt{5} - L_n + 2 < 2,$$

which yields

$$-\gamma < \psi(F_n\sqrt{5} - L_n + 2) < 1 - \gamma.$$

On the other hand if $2 \nmid n$ *then*

$$1 < F_n\sqrt{5} - L_n + 1 < 2,$$

which leads to

$$-\gamma < \psi(F_n\sqrt{5} - L_n + 1) < 1 - \gamma;$$

e) we have

$$\psi(\varphi^n) = \psi(F_n\varphi + F_{n-1}) = \psi\big((F_n\varphi - \lfloor F_n\varphi \rfloor + 1) + (\lfloor F_n\varphi \rfloor + F_{n-1} - 1)\big)$$

$$= \psi(F_n\varphi - \lfloor F_n\varphi \rfloor + 1) + \sum_{k=0}^{\lfloor F_n\varphi \rfloor + F_{n-1} - 2} \frac{1}{F_n\varphi - \lfloor F_n\varphi \rfloor + 1 + k} < 2,$$

and

$$1 < F_n\varphi - \lfloor F_n\varphi \rfloor + 1 = F_n\varphi - F_{n+1} + \frac{(-1)^n + 3}{2},$$

which implies that

$$-\gamma < \psi(F_n\varphi - \lfloor F_n\varphi \rfloor + 1) < 1 - \gamma;$$

f) $\psi(x^n) \sim n \log x$ *and*

$$x^n(\psi(x^n) - n\log x) \xrightarrow{n\to\infty} -\frac{1}{2}$$

for each $x > 1$. *In the sequel we obtain*

$$\psi(F_n\sqrt{5}) = \psi(\varphi^n - \bar{\varphi}^n) \sim n \log \varphi.$$

g)

$$\psi(F_n\sqrt{5}) = \begin{cases} \psi\left((\sqrt{5}-2)L_n - L_{n-3}+1\right) + \sum_{k=0}^{2L_n+L_{n-3}-2} \frac{1}{k+(\sqrt{5}-2)L_n - L_{n-3}+1} & \text{if } 2/n,\ n \geq 4, \\ \psi\left((\sqrt{5}-2)L_n - L_{n-3}+2\right) + \sum_{k=0}^{2L_n+L_{n-3}-3} \frac{1}{k+(\sqrt{5}-2)L_n - L_{n-3}+2} & \text{otherwise } (n \geq 5), \end{cases}$$

where

– *if* $2|n$ *then* $1 < (\sqrt{5}-2)L_n - L_{n-3}+1 < 2$ *and*

$$-\gamma < \psi\left((\sqrt{5}-2)L_n - L_{n-3}+1\right) < 1-\gamma,$$

– *if* $2 \nmid n$ *then* $1 < (\sqrt{5}-2)L_n - L_{n-3}+2 < 2$ *and*

$$-\gamma < \psi\left((\sqrt{5}-2)L_n - L_{n-3}+2\right) < 1-\gamma.$$

Proof.

a)

$$\psi(x+\varphi) - \psi\left(x-\frac{1}{\varphi}\right) = \psi(x+\varphi) - \psi(x+1-\varphi)$$

$$\overset{\text{by (2)}}{=} \sum_{k=1}^{\infty}\left(\frac{1}{x-\varphi+k} - \frac{1}{x+\varphi+k-1}\right)$$

$$= \sum_{k=1}^{\infty} \frac{2\varphi-1}{(x+k)^2-(x+k)-1} = \sum_{\tau=0}^{\infty} \frac{\sqrt{5}}{(x+\tau)^2+(x+\tau)-1}.$$

b)

$$\psi(\varphi^n) - \psi(1-\varphi^n) = -\pi\cot(\pi\varphi^n) = -\pi\cot(\pi F_n\varphi + \pi F_{n-1})$$

$$= -\pi\cot\pi F_n\varphi = \begin{cases} -\pi\cot\left(\pi\frac{F_n\sqrt{5}}{2}\right) & \text{if } 3|n, \\ \pi\tan\left(\pi\frac{F_n\sqrt{5}}{2}\right) & \text{if } 3\nmid n, \end{cases}$$

since $2|F_n \Leftrightarrow 3|n,\ n \in \mathbb{Z}$.

c)

$$\psi(\lfloor\varphi^n\rfloor) = \psi(\lfloor F_n\varphi + F_{n-1}\rfloor) = \psi(\lfloor F_n\varphi\rfloor + F_{n-1})$$
$$= \psi\left(F_{n+1} - \frac{(-1)^n + 1}{2} + F_{n-1}\right) = \psi\left(L_n - \frac{(-1)^n + 1}{2}\right)$$
$$= \begin{cases} -\gamma + H_{L_n-2} \text{if } 2|n, \\ -\gamma + H_{L_n+1} \text{if} 2 \nmid n. \end{cases}$$

e) the relations follow from the known identity (see also [16, 17]):

$$\varphi^n = F_n\varphi + F_{n-1}, \qquad n = 1,2,3, \ldots,$$

and identities (1) and (3).

f) There are constants C_1 (for instance $e^{-2\gamma}$) and C_2 (for instance 3^{-1}) (see [18] or [19]) such that

$$\frac{1}{2}\log(x^2 - x + C_1) \le \psi(x) \le \frac{1}{2}\log(x^2 - x + C_2)$$

for $x \in (1, \infty)$. Then for $x = y^n$ we get

$$y^n \log\frac{\sqrt{y^{2n} - y^n + C_1}}{y^n} \le y^n(\psi(y^n) - n\log y) \le y^n \log\frac{\sqrt{y^{2n} - y^n + C_2}}{y^n},$$

i.e.

$$\frac{1}{2}y^n\log(1 - y^{-n} + C_1y^{-2n}) \le y^n(\psi(y^n) - n\log y)$$
$$\le \frac{1}{2}y^n\log(1 - y^{-n} + C_2y^{-2n}).$$

Moreover, we have

$$y^n \log(1 - y^{-n}a_n) \to -1$$

when ever $a_n \xrightarrow{n\to\infty} 1$ which finishes the proof.

Corollary 4. *We have*

$$\psi(\varphi^2) = \psi(\varphi + 1) = \psi(\varphi) + \frac{1}{\varphi},$$

where $-\gamma < \psi(\varphi) < 1 - \gamma$ *since* $1 < \varphi < 2$,

$$\psi(\varphi^3) = \psi(2\varphi + 1) = \psi(2\varphi) + \frac{1}{2\varphi} = \psi(\sqrt{5}) + \frac{\sqrt{5}}{5} + \frac{1}{4}(\sqrt{5} - 1)$$
$$= \psi(\sqrt{5}) + \frac{9}{20}\sqrt{5} - \frac{1}{4} = \psi(\sqrt{5} - 1) + \frac{7}{10}\sqrt{5},$$

where $1 < \sqrt{5} - 1 < 2$, *i.e.* $-\gamma < \psi(\sqrt{5} - 1) < 1 - \gamma$,

$$\psi(\varphi^4) = \psi(3\varphi + 2) = \psi\left(\frac{3}{2}(\sqrt{5} - 1) + 5\right)$$
$$= \psi\left(\frac{3}{2}(\sqrt{5} - 1)\right) + \sum_{k=0}^{4} \frac{1}{\frac{3}{2}(\sqrt{5} - 1) + k}$$
$$= \psi\left(\frac{3}{2}(\sqrt{5} - 1)\right) - \frac{1}{2} + \frac{299}{66\sqrt{5}},$$

where $1 < \frac{3}{2}(\sqrt{5} - 1) < 2$, *i.e.* $-\gamma < \psi\left(\frac{3}{2}(\sqrt{5} - 1)\right) < 1 - \gamma$.

5. Final remark

In the previous theorem, arguments of the psi function were reduced to numbers contained in the interval $(1,2)$. This operation is especially interesting from the algebraic point of view, but it is not proper from the numerical point of view (asimilar situation takes place for the gamma function - it is better to replace a given argument by the respective large value of the argument, with respect to the Stirling formula, which plays not only an asymptotic formula role but even the exact role in approximation of values of the gamma function). The values of the psi function could be calculated using the following three inequalities of the asymptotic type which hold true for all $x > 1$ (and which especially play a crucial role for large values of x):

$$\log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log x - \frac{1}{2x} - \frac{2\gamma - 1}{2x^2}$$

(see [20, Theorem C]),

$$\frac{1}{2}\log(x^2 + x + e^{-2\gamma}) \le \psi(x + 1) \le \frac{1}{2}\log\left(x^2 + x + \frac{1}{3}\right)$$

(see [18]),

$$\log\left(x - \frac{1}{\sqrt{6}}\right) - \frac{2}{(6 + 2\sqrt{6})x} \le \psi(x) \le \log\left(x + \frac{1}{\sqrt{6}}\right) - \frac{1}{(6 - 2\sqrt{6})x}$$

(see [21]), or using the asymptotic series of function$\psi(x)$as $x \to \infty$:

$$\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}$$

(see [5] and [6]).

6. Conclusions

In the paper, several formulae for values of the psi function are presented. Some of them are of theasymptotic type, and some of them are of the symbolic-arithmetic type. It seems that the planned goal has been accomplished, and even more (as in the case of the integral formula for the psi function for powers of argument which, from a numerical point of view, is better than the respective Legendre's integral formula). It is important that in comparison with many formulae for the psi function (also recently proven), relations obtained in the present paper complement the existing directory of these formulae well and substantially.

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