

SOLUTIONS OF SOME FUNCTIONAL EQUATIONS IN A CLASS OF GENERALIZED HÖLDER FUNCTIONS

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Abstract. The existence and uniqueness of solutions a nonlinear iterative equation in the class of r -times differentiable functions with the r -derivative satisfying a generalized Hölder condition is considered.

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1. Introduction

In [1, 2] the space $W_\gamma[a, b]$ ($W_\gamma^r[a, b]$) of r times differentiable functions with the r -the derivative satisfying generalized γ -Hölder condition was introduced and some of its properties proved. In the present paper we examine the existence and uniqueness of solutions of a nonlinear iterative functional equation in this class of functions. We apply some ideas from Kuczma [3], Matkowski [4, 5] (see also Kuczma, Choczewski, Ger [6]), where differentiable solutions, Lipschitzian solutions, bounded variation solutions of different type of iterative functional equations were investigated.

2. Preliminaries

Consider non-linear functional equation

$$\varphi(x) = h(\varphi[f(x)]) + g(x) \quad (1)$$

where f, g, h are given and φ is a unknown function.

We accept the following notation: $I = [a, b]$, $a, b \in R$, $d := b - a$, $W_\gamma(I)$ - is the Banach space of the r -time differentiable functions defined on the interval I with values in R , such that, for some $M \geq 0$; its r -th derivative satisfies the following γ -Hölder condition

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \leq M\gamma(|x - \bar{x}|), \quad \bar{x}, x \in I.$$

where a fixed function γ satisfies the following condition (see [1, 2]):

$$(I) \gamma: [0, d] \rightarrow [0, \infty) \text{ is increasing and concave, } \gamma(0) = 0, \lim_{t \rightarrow 0^+} \gamma(t) = \gamma(0), \\ \lim_{t \rightarrow d^-} \gamma(t) = \gamma(d), \gamma'_+(0) = +\infty$$

We assume that

- (i) $f: I \rightarrow I, f \in W_\gamma(I), \sup_I |f'| \leq 1$
- (ii) $g: I \rightarrow R, g \in W_\gamma(I)$
- (iii) $h: R \rightarrow R, h \in C^r, h^{(r)}$ fulfils the Lipschitz condition in R .
- (iv) there exists $\xi \in I$ such that $\lim_{n \rightarrow \infty} f^n(x) = \xi, x \in I$, where f^n is the n -th iteration function f
- (v) is analytic function at η_0 , where η_0 is the solution of equation $\eta_0 = h(\eta_0) + g(\xi)$

We define functions $h_k: I \times R^{k+1} \rightarrow R, k = 0, 1, \dots, r-1$ by the formula

$$\begin{cases} h_0(x, y_0) := h(y_0) + g(x) \\ h_{k+1}(x, y_0, \dots, y_{k+1}) := \frac{\partial h_k}{\partial x} + f'(x) \left(\frac{\partial h_k}{\partial y_0} y_1 + \dots + \frac{\partial h_k}{\partial y_k} y_{k+1} \right). \end{cases} \quad (2)$$

Lemma 1. [4]

By assumptions (i)-(iii), h_k defined by (2) are of the form:

1. for $r = 1$

$$h_1(x, y_0, y_1) = h'(y_0)y_1 f'(x) + g'(x); \quad (3)$$

2. for $r \geq 2, k = 2, \dots, r$

$$h_k(x, y_0, \dots, y_k) = p_k(x, y_0, \dots, y_{k-1}) + h'(y_0)y_k (f'(x))^k + \\ + h'(y_0)y_1 f^{(k)}(x) + g^{(k)}(x), \quad (4)$$

where

$$p_k(x, y_0, \dots, y_{k-1}) + h'(y_0)y_k (f'(x))^k = \\ = \sum_{i=1}^k h^{(k-i+1)}(y_0) \sum_{\alpha_1 + \dots + \alpha_i = k-i+1} u_{\alpha_1 \dots \alpha_i, k}(x) y_1^{\alpha_1} \dots y_i^{\alpha_i} \quad (5)$$

and $u_{\alpha_1 \dots \alpha_i, k}(x)$ are of the class C^{r-k+1} in I , for all numbers $\alpha_1, \dots, \alpha_i \in N$ such that $\alpha_1 + \dots + \alpha_i = k - i + 1, k = 2, \dots, r, i = 1, \dots, k$.

Remark 1.

If (i)-(iii) are fulfilled, then $h_r: I \times R^{k+1} \rightarrow R$, given by

$$h_r(x, y_0, \dots, y_r) = h'(y_0)y_1 f^{(r)}(x) + g^{(r)}(x) + \\ + \sum_{i=1}^r h^{(r-i+1)}(y_0) \sum_{\alpha_1 + \dots + \alpha_i = r-i+1} u_{\alpha_1 \dots \alpha_i, r}(x) y_1^{\alpha_1} \dots y_i^{\alpha_i}$$

fulfill γ -Hölder condition for $x \in I$ and Lipschitz condition with respect to y_i , $i = 0, \dots, r$ in $Z := [a_0, b_0] \times [a_1, b_1] \times \dots \times [a_r, b_r]$. It means, that there are positive constants m, l_0, \dots, l_{r-1} and

$$l_r = \sup_{I \times [a_0, b_0]} |h'(f')^r|,$$

such that for $(x, y_1, \dots, y_r), (\bar{x}, \bar{y}_1, \dots, \bar{y}_r) \in Z$ we have

$$|h_r(x, y_0, \dots, y_r) - h_r(\bar{x}, \bar{y}_0, \dots, \bar{y}_r)| \leq m\gamma(|x - \bar{x}|) + l_0|y_0 - \bar{y}_0| + \dots + l_r|y_r - \bar{y}_r|.$$

Define the functions $w_{r,i}: I \times R^i \rightarrow R$, $i = 1, 2, \dots, r$ by the following formulas:

$$w_{r,i}(x, y_1, \dots, y_i) := \sum_{\alpha_1 + \dots + \alpha_i = r-i+1} u_{\alpha_1 \dots \alpha_i, r}(x) y_1^{\alpha_1} \dots y_i^{\alpha_i}. \quad (6)$$

Remark 2.

The functions $w_{r,i}$ defined by (6) fulfill γ -Hölder condition with respect to variable x in I and Lipschitz condition with respect to the variable y_i , $i = 1, \dots, r$ in each set $Z_i := [a_1, b_1] \times \dots \times [a_i, b_i]$.

Remark 3.

If f, g, h satisfy the assumptions (i)-(iii) and $\varphi \in W_\gamma(I)$ is a solution of equation (1) then the derivatives $\varphi^{(k)}$, $k = 0, \dots, r$ satisfy the system of equations

$$\varphi^{(k)}(x) = h_k(x, \varphi[f(x)], \dots, \varphi^{(k)}[f(x)]), \quad x \in I.$$

If assumptions (i)-(iv) are fulfilled and $\varphi \in W_\gamma(I)$ is a solution of equation (1) in I , then the numbers

$$\eta_k = \varphi^{(k)}(\xi), \quad k = 0, \dots, r \quad (7)$$

satisfy the system of equations

$$\eta_k = h_k(\xi, \eta_0, \dots, \eta_k), \quad k = 0, \dots, r, \quad (8)$$

where h_k are defined by (2).

Remark 4.

Let $\varphi \in W_\gamma(I)$ be a solution of the equation (1). Present φ in the following form

$$\varphi(x) = P(x) + \psi(x - \xi), \quad x \in I = [a, b] \quad (9)$$

where $\psi: [a - \xi, b - \xi] \rightarrow R$ and $P(x) = \sum_{i=0}^r \frac{\eta_i}{i!} (x - \xi)^i$, $x \in [a, b]$.

Define the functions

$$\bar{f}(x) := f(x + \xi) - \xi, \quad x \in [a - \xi, b - \xi]$$

$$\bar{g}(x) := g(x + \xi), \quad x \in [a - \xi, b - \xi]$$

and for $y \in R$, $x \in [a - \xi, b - \xi]$

$$\bar{h}(x) := h(P[f(x + \xi)] + y) - P(x + \xi).$$

It follows from above definitions and equation (9) that ψ satisfies the following equation

$$\psi(x) = \bar{h}(\psi[\bar{f}(x)]) + \bar{g}(x), x \in [a - \xi, b - \xi].$$

It is easy to prove, that if assumptions (i)-(iv) are fulfilled and $\eta_i, i = 0, \dots, r$, are the solution of equations (8), then the function $\varphi \in W_\gamma[a, b]$ satisfies the equation (1) in $[a, b]$ and the condition (7) if and only if the function ψ given by (9) belongs to $W_\gamma[a - \xi, b - \xi]$ and satisfies

$$\psi^{(k)}(0) = 0, k = 0, \dots, r.$$

Thus, we assume that $0 \in I$ and consider the equation (1) whose solution satisfies the condition

$$\varphi^{(k)}(0) = 0, k = 0, \dots, r.$$

Then system of equations (8) takes the following form

$$h_k(0, \dots, 0) = 0, k = 0, \dots, r.$$

3. Main result

Theorem 1.

If assumptions (i)-(iii) are fulfilled, f is a monotone function in the interval I , the conditions (iv) and (v) are fulfilled for $\xi = 0$, $\eta_0 = 0$ and

$$h_k(0, \dots, 0) = 0, \quad k = 1, \dots, r; \tag{10}$$

$$|h'(0)(f'(0))^r| < 1 \tag{11}$$

then equation (1) has exactly one solution $\varphi \in W_\gamma(I)$ satisfying the condition

$$\varphi^{(k)}(0) = 0, k = 0, \dots, r. \tag{12}$$

Moreover, there exists a neighbourhood U of the point $\xi = 0$ and the number r_0 such that for a function $\varphi_0 \in W_\gamma(\bar{U})$, satisfying the condition (12) and the inequality $\|\varphi_0\| \leq r_0$, a sequence of functions

$$\varphi_n(x) = h(\varphi_{n-1}[f(x)]) + g(x), \quad x \in \bar{U},$$

converges to a solution of (1) according to the norm in the space $W_\gamma(\bar{U})$.

Proof.

From (v) we have $h(y) = \sum_{n=0}^{\infty} a_n y^n$ in some neighbourhood of the point 0. Denote by R_0 the radius of convergence of this series. From (11) and from the continuity of functions $(f')^r$ and h' , from definition of the function γ there exists a neighbourhood V of the point $\xi = 0$ and $d < R_0, 0 < \theta < 1$ such that

$$\sup_{\bar{V} \times [-d, d]} |h'(f')^r| \leq \theta, f(V) \subset V, \gamma(\text{diam } \bar{V}) \geq \text{diam } \bar{V}. \quad (13)$$

From Remark 1, definition of γ and from (13) there are positive constants m, l_0, \dots, l_{r-1} , and $l_r = \theta$, that in $\bar{V} \times [-d, d]^{r+1}$ we have

$$\begin{aligned} |h_r(x, y_0, \dots, y_r) - h_r(\bar{x}, \bar{y}_0, \dots, \bar{y}_r)| &\leq m\gamma(|x - \bar{x}|) + l_0|y_0 - \bar{y}_0| + \dots + \\ &+ \theta|y_r - \bar{y}_r|. \end{aligned} \quad (14)$$

From Remark 2, definition of γ there are in $Z_i = \bar{V} \times [-d, d]^i$ constants $B_{i,0}, B_{i,k}$, $i = 1, \dots, r$, $k = 1, \dots, i$, such that

$$|w_{r,i}(x, y_1, \dots, y_i) - w_{r,i}(\bar{x}, \bar{y}_1, \dots, \bar{y}_i)| \leq B_{i,0}\gamma(|x - \bar{x}|) + \sum_{k=1}^i B_{i,k}|y_k - \bar{y}_k| \quad (15)$$

We accept the following notation:

$$W_i := \sup_{\bar{V} \times [-d, d]} |w_{r,i}|, \quad i = 1, 2, \dots, r; \quad (16)$$

$$H_i := \sup_{\bar{V} \times [-d, d]} |h^{(i)}|, \quad i = 1, 2, \dots, r + 1; \quad (17)$$

$$F := \sup_{\bar{V}} |f^{(r)}|; K \text{ is a } \gamma\text{-Hölder constant of } f^{(r)} \text{ in } \bar{V}; \quad (18)$$

$$C_{\alpha_1 \dots \alpha_i, r} := \sup_{\bar{V}} |u_{\alpha_1 \dots \alpha_i, r}|, \quad i = 1, 2, \dots, r, \quad \alpha_1 + \dots + \alpha_i = r - i + 1; \quad (19)$$

$$D_{\alpha_1 \dots \alpha_i, r} := \sup_{\bar{V}} |u'_{\alpha_1 \dots \alpha_i, r}|, \quad i = 1, 2, \dots, r, \quad \alpha_1 + \dots + \alpha_i = r - i + 1. \quad (20)$$

By $\sum a_{\alpha_1 \dots \alpha_i r}$ we denote the sum of $a_{\alpha_1 \dots \alpha_i r}$ for all $\alpha_1, \dots, \alpha_i \in N$ such that $\alpha_1 + \dots + \alpha_i = r - i + 1$, $i = 1, 2, \dots, r$.

In view of Lemma 1, we have

$$u_{0 \dots 0 1_i r} = (f')^r$$

and, from (13), we get

$$|h'(y)u_{0 \dots 0 1_i r}(x)| \leq \theta, \quad x \in \bar{V}, y \in [-d, d] \quad (21)$$

Let us take $c_1 \in (0, b - a]$, $c_1 \leq \gamma(c_1) \leq 1$ and

$$\gamma(c_1) \sum_{i=0}^{r-1} l_i c_1^{r-i-1} < 1 - \theta.$$

Put

$$r_0 := \frac{m}{1 - \theta - \gamma(c_1) \sum_{i=0}^{r-1} l_i c_1^{r-i-1}}. \quad (22)$$

Then let's take $c_2 \in (0, b - a]$ such that $c_2 \leq \gamma(c_2) \leq \min\{1, \frac{d}{r_0}\}$ and

$$\begin{aligned} l_0 := & H_1 F(\gamma(c_2))^{r-1} + H_2 F(\gamma(c_2))^{2r} + H_1 K(\gamma(c_2))^r + H_2 F r_0 (\gamma(c_2))^{2r} + \\ & + H_2 K r_0 (\gamma(c_2))^{2r+1} + F r_0 (\gamma(c_2))^{2r} \sum_{n=2}^{\infty} n(n-1)^2 |a_n| r_0^{n-2} (\gamma(c_2))^{(n-2)(r+1)} \\ & + (\gamma(c_2))^r \sum_{i=1}^r W_i \sum_{n=r-i+2}^{\infty} |a_n| n(n-1)(n-r+i-2)^2 r_0^{n-r+i-2} \\ & \cdot (\gamma(c_2))^{(n-r+i-2)(r+1)} + \\ & + (\gamma(c_2))^{r+1} \left(\sum_{i=1}^r H_{r-i+2} \left(B_{i,0} + 2r_0 \sum_{i=1}^i B_{i,k} (\gamma(c_2))^{r-k} \right) \right) + \\ & + \sum_{i=1}^{r-1} H_{r-i+1} \sum C_{\alpha_1 \dots \alpha_i r} r_0^{r-i} (\gamma(c_2))^{r\alpha_1 + \dots + (r-i+1)\alpha_i - 1} (r-i+1)^2 + \\ & + H_1 \sum C_{\alpha_1 \dots \alpha_{r-1} 0, r} (\gamma(c_2))^{r\alpha_1 + (r-1)\alpha_2 + \dots + 2\alpha_{r-1} - 1} + \\ & + \sum_{i=1}^r H_{r-i+1} r_0^{r-i} (r-i+1) \sum D_{\alpha_1 \dots \alpha_i r} (\gamma(c_2))^{r\alpha_1 + \dots + (r-i+1)\alpha_i} < 1 - \theta. \quad (23) \end{aligned}$$

Choose $c \leq \min\{c_1, c_2\}$. Of course $c \leq \gamma(c) \leq \frac{d}{r_0}$. We will select a neighborhood of zero $U \subset V$ such that $f(U) \subset U$ and $\text{diam} \bar{U} \leq c$.

Consider the Banach space $W_\gamma(\bar{U})$ with the norm:

$$\|\varphi\| := \sum_{k=0}^r |\varphi^{(k)}(0)| + \sup \left\{ \frac{|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})|}{\gamma(|x - \bar{x}|)}; x, \bar{x} \in \bar{U}, x \neq \bar{x} \right\}.$$

Let us define the set

$$A_{r_0} := \{ \varphi \in W_\gamma(\bar{U}), \varphi^{(k)}(0) = 0, k = 0, \dots, r, \|\varphi\| \leq r_0 \}.$$

Note that A_{r_0} is a closed subset of Banach space $W_\gamma(\bar{U})$ and for $\varphi \in A_{r_0}$ the norm is expressed by the formula

$$\|\varphi\| := \sup \left\{ \frac{|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})|}{\gamma(|x - \bar{x}|)}; x, \bar{x} \in \bar{U}, x \neq \bar{x} \right\} \quad (24)$$

Thus, the set A_{r_0} with the metric $\varrho(\varphi_1, \varphi_2) := \|\varphi_1 - \varphi_2\|$ is a complete metric space.

By the mean value theorem and by definition of the number of c we have for $\varphi \in A_{r_0}$

$$\sup |\varphi^{(k)}| \leq c^{r-k} \gamma(c) r_0 \leq \gamma(c) r_0 \leq d, \quad k = 0, \dots, r \quad (25)$$

and so $\varphi^{(k)} \in [-d, d]$, $k = 0, \dots, r$.

For $\varphi \in A_{r_0}$ define the transformation T by the formula

$$(T\varphi)(x) := h(\varphi[f(x)]) + g(x), \quad x \in \bar{U}.$$

We will show that $T(A_{r_0}) \subset A_{r_0}$.

Based on Remarks 1 and 3 the function $\psi := T\varphi$ belongs to $W_\gamma(\bar{U})$, from (iv) and (10), (12) appears that $\psi^{(k)}(0) = 0$, $k = 0, \dots, r$. Then using the formulas (12), (13), (22), (25) and the assumption (i) we obtain

$$\begin{aligned} |\psi^{(r)}(x) - \psi^{(r)}(\bar{x})| &\leq m\gamma(|x - \bar{x}|) + l_0 |\varphi[f(x)] - \varphi[f(\bar{x})]| + \dots + \\ &+ l_{r-1} |\varphi^{(r-1)}[f(x)] - \varphi^{(r-1)}[f(\bar{x})]| + \theta |\varphi^{(r)}[f(x)] - \varphi^{(r)}[f(\bar{x})]| \leq \\ &(m + l_0 c^{r-1} \gamma(c) r_0 + \dots + l_{r-1} \gamma(c) r_0 + \theta r_0) \gamma(|x - \bar{x}|) \leq r_0 \gamma(|x - \bar{x}|). \end{aligned}$$

Which means from (24) that $\|T\varphi\| \leq r_0$. Thus $T(A_{r_0}) \subset A_{r_0}$.

Now we prove that T is a contraction map. Let us put $\psi_1 := T\varphi_1$, $\psi_2 := T\varphi_2$. Basing on formulas (4)-(5) of Lemma 1 and from (24) we have

$$\begin{aligned}
& \left| \psi_1^{(r)}(x) - \psi_1^{(r)}(\bar{x}) - \psi_2^{(r)}(x) + \psi_2^{(r)}(\bar{x}) \right| = \\
& = \left| h'(\varphi_1[f(x)])\varphi_1'[f(x)]f^{(r)}(x) - h'(\varphi_1[f(\bar{x})])\varphi_1'[f(\bar{x})]f^{(r)}(\bar{x}) + \right. \\
& - h'(\varphi_2[f(x)])\varphi_2'[f(x)]f^{(r)}(x) + h'(\varphi_2[f(\bar{x})])\varphi_2'[f(\bar{x})]f^{(r)}(\bar{x}) + \\
& + \sum_{i=1}^r \left(h^{(r-i+1)}(\varphi_1[f(x)])w_{r,i}(x, \varphi_1'[f(x)], \dots, \varphi_1^{(i)}[f(x)]) + \right. \\
& \quad \left. - h^{(r-i+1)}(\varphi_1[f(\bar{x})])w_{r,i}(\bar{x}, \varphi_1'[f(\bar{x})], \dots, \varphi_1^{(i)}[f(\bar{x})]) + \right. \\
& \quad \left. - h^{(r-i+1)}(\varphi_2[f(x)])w_{r,i}(x, \varphi_2'[f(x)], \dots, \varphi_2^{(i)}[f(x)]) + \right. \\
& \quad \left. + h^{(r-i+1)}(\varphi_2[f(\bar{x})])w_{r,i}(\bar{x}, \varphi_2'[f(\bar{x})], \dots, \varphi_2^{(i)}[f(\bar{x})]) \right) \leq \\
& \left| h'(\varphi_1[f(x)]) \right| \left| f^{(r)}(x) \right| \left| \varphi_1'[f(x)] - \varphi_1'[f(\bar{x})] - \varphi_2'[f(x)] + \varphi_2'[f(\bar{x})] \right| + \\
& + \left| f^{(r)}(x) \right| \left| \varphi_1'[f(\bar{x})] - \varphi_2'[f(\bar{x})] \right| \left| h'(\varphi_1[f(x)]) - h'(\varphi_1[f(\bar{x})]) \right| + \\
& + \left| h'(\varphi_1[f(\bar{x})]) \right| \left| \varphi_1'[f(\bar{x})] - \varphi_2'[f(\bar{x})] \right| \left| f^{(r)}(x) - f^{(r)}(\bar{x}) \right| + \\
& + \left| f^{(r)}(x) \right| \left| \varphi_2'[f(x)] - \varphi_2'[f(\bar{x})] \right| \left| h'(\varphi_1[f(x)]) - h'(\varphi_2[f(x)]) \right| + \\
& + \left| \varphi_2[f(\bar{x})] \right| \left| f^{(r)}(x) - f^{(r)}(\bar{x}) \right| \left| h'(\varphi_1[f(x)]) - h'(\varphi_2[f(x)]) \right| + \\
& + \left| \varphi_2[f(\bar{x})] \right| \left| f^{(r)}(\bar{x}) \right| \left| h'(\varphi_1[f(x)]) - h'(\varphi_2[f(x)]) - h'(\varphi_1[f(\bar{x})]) + h'(\varphi_2[f(\bar{x})]) \right| + \\
& + \sum_{i=1}^r \left(\left| w_{r,i}(x, \varphi_1'[f(x)], \dots, \varphi_1^{(i)}[f(x)]) \right| \left| h^{(r-i+1)}(\varphi_1[f(x)]) - h^{(r-i+1)}(\varphi_1[f(\bar{x})]) + \right. \right. \\
& \quad \left. \left. - h^{(r-i+1)}(\varphi_2[f(x)]) + h^{(r-i+1)}(\varphi_2[f(\bar{x})]) \right| + \right. \\
& \quad \left. + \left| h^{(r-i+1)}(\varphi_1[f(\bar{x})]) - h^{(r-i+1)}(\varphi_2[f(\bar{x})]) \right| \cdot \right. \\
& \quad \left. \cdot \left| w_{r,i}(x, \varphi_1'[f(x)], \dots, \varphi_1^{(i)}[f(x)]) - w_{r,i}(\bar{x}, \varphi_1'[f(\bar{x})], \dots, \varphi_1^{(i)}[f(\bar{x})]) \right| + \right. \\
& \quad \left. + \left| h^{(r-i+1)}(\varphi_2[f(x)]) - h^{(r-i+1)}(\varphi_2[f(\bar{x})]) \right| \cdot \right. \\
& \quad \left. \cdot \left| w_{r,i}(x, \varphi_1'[f(x)], \dots, \varphi_1^{(i)}[f(x)]) - w_{r,i}(x, \varphi_2'[f(x)], \dots, \varphi_2^{(i)}[f(x)]) \right| + \right. \\
& + \left| h^{(r-i+1)}(\varphi_2[f(\bar{x})]) \right| \left| w_{r,i}(x, \varphi_1'[f(x)], \dots, \varphi_1^{(i)}[f(x)]) - w_{r,i}(\bar{x}, \varphi_1'[f(\bar{x})], \dots, \varphi_1^{(i)}[f(\bar{x})]) + \right. \\
& \quad \left. - w_{r,i}(x, \varphi_2'[f(x)], \dots, \varphi_2^{(i)}[f(x)]) + w_{r,i}(x, \varphi_2'[f(\bar{x})], \dots, \varphi_2^{(i)}[f(\bar{x})]) \right|.
\end{aligned}$$

Note, that if $\varphi_1, \varphi_2 \in A_{r_0}$, then in view of the mean value theorem, from the definition of the number c and from (i) we have the following inequalities

$$\sup_{\bar{c}} \left| \varphi_i^{(k)} \right| \leq r_0 c^{r-k} \gamma(c) \leq r_0 (\gamma(c))^{r-k+1}, \quad k = 0, \dots, r, \quad i = 1, 2; \quad (26)$$

$$\left| \varphi_1^{(k)} [f(x)] - \varphi_1^{(k)} [f(\bar{x})] \right| \leq r_0 (\gamma(c))^{r-k} \gamma(|x - \bar{x}|), \quad k = 0, \dots, r, \quad x, \bar{x} \in \bar{U}; \quad (27)$$

$$\left| \varphi_1^{(k)} [f(x)] - \varphi_2^{(k)} [f(x)] \right| \leq \|\varphi_1 - \varphi_2\| (\gamma(c))^{r-k+1}, \quad k = 0, \dots, r, \quad x \in \bar{U}; \quad (28)$$

$$\left| \varphi_1^{(k)} [f(x)] - \varphi_1^{(k)} [f(\bar{x})] - \varphi_2^{(k)} [f(x)] + \varphi_2^{(k)} [f(\bar{x})] \right| \leq \|\varphi_1 - \varphi_2\| (\gamma(c))^{r-k} \gamma(|x - \bar{x}|), \quad (29)$$

$k = 0, \dots, r, \quad x, \bar{x} \in \bar{U}.$

By induction on $l \in N$ we also obtain:

$$\left| \left(\varphi_1^{(k)} [f(x)] \right)^l - \left(\varphi_1^{(k)} [f(\bar{x})] \right)^l - \left(\varphi_2^{(k)} [f(x)] \right)^l + \left(\varphi_2^{(k)} [f(\bar{x})] \right)^l \right| \leq \quad (30)$$

$$l^2 r_0^{l-1} (\gamma(c))^{l(r-k)+l-1} \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|), \quad k = 0, \dots, r, \quad x, \bar{x} \in \bar{U}, \quad l = 1, 2, \dots$$

From (v) and by selection of d we have uniform and absolute convergence of the series

$$h'(y) = \sum_{n=1}^{\infty} n a_n y^{n-1} \quad \text{for } y \in [-d, d].$$

Let's consider the expression:

$$\left| h'(\varphi_1 [f(x)]) - h'(\varphi_1 [f(\bar{x})]) - h'(\varphi_2 [f(x)]) + h'(\varphi_2 [f(\bar{x})]) \right| =$$

$$= \left| \sum_{n=2}^{\infty} n a_n \left(\left(\varphi_1 [f(x)] \right)^{n-1} - \left(\varphi_1 [f(\bar{x})] \right)^{n-1} - \left(\varphi_2 [f(x)] \right)^{n-1} + \left(\varphi_2 [f(\bar{x})] \right)^{n-1} \right) \right|.$$

From (30) we obtain

$$\left| \left(\varphi_1 [f(x)] \right)^{n-1} - \left(\varphi_1 [f(\bar{x})] \right)^{n-1} - \left(\varphi_2 [f(x)] \right)^{n-1} + \left(\varphi_2 [f(\bar{x})] \right)^{n-1} \right| \leq$$

$$\leq (n-1)^2 r_0^{n-2} (\gamma(c))^{(n-1)r+n-2} \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|), \quad x, \bar{x} \in \bar{U}, \quad n = 2, 3, \dots$$

Note that a series

$$\sum_{n=2}^{\infty} A_n \quad \text{where } A_n := n |a_n| (n-1)^2 r_0^{n-2} (\gamma(c))^{(n-1)r+n-2}$$

converges, because the numbers c, d have been selected in such a way that

$$\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \frac{r_0}{R_0} (\gamma(c))^{r+1} \leq \frac{r_0 \gamma(c)}{R_0} \leq \frac{d}{R_0} < 1.$$

Therefore

$$\begin{aligned} & \left| h'(\varphi_1[f(x)]) - h'(\varphi_1[f(\bar{x})]) - h'(\varphi_2[f(x)]) + h'(\varphi_2[f(\bar{x})]) \right| \leq \\ & \leq \sum_{n=2}^{\infty} n(n-1)^2 |a_n| r_0^{n-2} (\gamma(c))^{(n-1)r+n-2} \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|), \quad x, \bar{x} \in \bar{U}. \end{aligned} \quad (31)$$

Similarly for $x, \bar{x} \in \bar{U}$, $i = 1, \dots, r$ we get

$$\begin{aligned} & \left| h^{(r-i+1)}(\varphi_1[f(x)]) - h^{(r-i+1)}(\varphi_1[f(\bar{x})]) - h^{(r-i+1)}(\varphi_2[f(x)]) + h^{(r-i+1)}(\varphi_2[f(\bar{x})]) \right| \leq \\ & \sum_{n=r-i+2}^{\infty} |a_n| n \dots (n-r+i)(n-r+i-1)^2 r_0^{n-r+i-2} (\gamma(c))^{(n-r+i-1)r+n-r+i-2} \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|). \end{aligned} \quad (32)$$

By induction and from (26)-(29) we have

$$\begin{aligned} & \left| (\varphi_1'[f(x)])^{\alpha_1} \dots (\varphi_1^{(i)}[f(x)])^{\alpha_i} - (\varphi_2'[f(x)])^{\alpha_1} \dots (\varphi_2^{(i)}[f(x)])^{\alpha_i} \right| \leq \\ & \leq (\alpha_1 + \dots + \alpha_i) r_0^{\alpha_1 + \dots + \alpha_i - 1} (\gamma(c))^{r\alpha_1 + \dots + (r-i+1)\alpha_i} \|\varphi_1 - \varphi_2\|, \\ & \alpha_1, \dots, \alpha_i \in N, \quad i = 1, \dots, r, \quad x, \bar{x} \in \bar{U}, \varphi_1, \varphi_2 \in A_{r_0} \end{aligned} \quad (33)$$

$$\begin{aligned} & \left| (\varphi_2'[f(x)])^{\alpha_1} \dots (\varphi_2^{(i)}[f(x)])^{\alpha_i} - (\varphi_2'[f(\bar{x})])^{\alpha_1} \dots (\varphi_2^{(i)}[f(\bar{x})])^{\alpha_i} \right| \leq \\ & \leq (\alpha_1 + \dots + \alpha_i) r_0^{\alpha_1 + \dots + \alpha_i - 1} (\gamma(c))^{r\alpha_1 + \dots + (r-i+1)\alpha_i - 1} \gamma(|x - \bar{x}|), \\ & i = 1, \dots, r, \quad x, \bar{x} \in \bar{U}, \quad \varphi_2 \in A_{r_0}. \end{aligned} \quad (34)$$

Now from (33) and (34) we get

$$\begin{aligned} & \left| (\varphi_1'[f(x)])^{\alpha_1} \dots (\varphi_1^{(i)}[f(x)])^{\alpha_i} - (\varphi_1'[f(\bar{x})])^{\alpha_1} \dots (\varphi_1^{(i)}[f(\bar{x})])^{\alpha_i} + \right. \\ & \left. - (\varphi_2'[f(x)])^{\alpha_1} \dots (\varphi_2^{(i)}[f(x)])^{\alpha_i} + (\varphi_2'[f(\bar{x})])^{\alpha_1} \dots (\varphi_2^{(i)}[f(\bar{x})])^{\alpha_i} \right| \leq \\ & \leq (\alpha_1 + \dots + \alpha_i)^2 r_0^{\alpha_1 + \dots + \alpha_i - 1} (\gamma(c))^{r\alpha_1 + \dots + (r-i+1)\alpha_i - 1} \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|), \\ & i = 1, \dots, r, \quad x, \bar{x} \in \bar{U}, \quad \varphi_1, \varphi_2 \in A_{r_0}. \end{aligned} \quad (35)$$

From (6), by the mean value theorem and from (33) and (34) we get

$$\begin{aligned}
& \left| w_{r,i}(x, \varphi_1' [f(x)], \dots, \varphi_1^{(i)} [f(x)]) - w_{r,i}(\bar{x}, \varphi_1' [f(\bar{x})], \dots, \varphi_1^{(i)} [f(\bar{x})]) + \right. \\
& \left. - w_{r,i}(x, \varphi_2' [f(x)], \dots, \varphi_2^{(i)} [f(x)]) + w_{r,i}(\bar{x}, \varphi_2' [f(\bar{x})], \dots, \varphi_2^{(i)} [f(\bar{x})]) \right| \leq \\
& \leq \sum |u_{\alpha_1 \dots \alpha_i, r}(x)| (r-i+1)^2 r_0^{r-i} (\gamma(c))^{r\alpha_1 + \dots + (r-i+1)\alpha_i - 1} \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|) + \quad (36) \\
& + \sum |u_{\alpha_1 \dots \alpha_i, r}(z)| (r-i+1) r_0^{r-i} (\gamma(c))^{r\alpha_1 + \dots + (r-i+1)\alpha_i} \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|) \\
& \quad i = 1, 2, \dots, r, x, \bar{x} \in \bar{U}, z \text{ is between } x \text{ and } \bar{x}.
\end{aligned}$$

Now, from (15)-(22), (27)-(32) and (36) we get

$$\begin{aligned}
& \left| \psi_1^{(r)}(x) - \psi_1^{(r)}(\bar{x}) - \psi_2^{(r)}(x) + \psi_2^{(r)}(\bar{x}) \right| \leq \\
& \leq (H_1 F(\gamma(c)))^{r-1} + H_2 F r_0 (\gamma(c))^{2r} + H_1 K (\gamma(c))^r + \\
& \quad + H_2 F r_0 (\gamma(c))^{2r} + H_2 K r_0 (\gamma(c))^{2r+1} + \\
& \quad + F r_0 (\gamma(c))^{2r} \sum_{n=2}^{\infty} |a_n| n(n-1)^2 r_0^{n-2} (\gamma(c))^{(n-2)(r+1)} + \\
& \quad + \sum_{i=1}^r W_i (\gamma(c))^r \sum_{n=r-i+2}^{\infty} |a_n| n(n-1) \dots (n-r+i)(n-r+i-1)^2 r_0^{n-r+i-2} \cdot \\
& \cdot (\gamma(c))^{(n-r+i-2)(r-1)r} + (\gamma(c))^{r+1} \left(\sum_{i=1}^r H_{r-i+2} \left(B_{i,0} + 2r_0 \sum_{k=1}^i B_{i,k} (\gamma(c))^{r-k} \right) \right) + \\
& \quad + \sum_{i=1}^{r-1} H_{r-i+1} \sum C_{\alpha_1 \dots \alpha_i, r} (r-i+1)^2 r_0^{r-i} (\gamma(c))^{r\alpha_1 + \dots + (r-i+1)\alpha_i - 1} + \\
& \quad + H_1 \sum_{\alpha_1 + \dots + \alpha_{r-1} = 1, \alpha_r = 0} C_{\alpha_1 \dots \alpha_{r-1}, 0, r} (\gamma(c))^{r\alpha_1 + \dots + 2\alpha_{r-1} - 1} + \\
& \quad + \sum_{i=1}^{r-1} H_{r-i+1} \sum D_{\alpha_1 \dots \alpha_i, r} (r-i+1) r_0^{r-i} (\gamma(c))^{r\alpha_1 + \dots + (r-i+1)\alpha_i} + \\
& \quad + \sup_{\bar{U}} |h' u_{0 \dots 01, r}| \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|) \leq (l_0 + \theta) \|\varphi_1 - \varphi_2\| \gamma(|x - \bar{x}|).
\end{aligned}$$

Putting $L = l_0 + \theta$ and making use of definition (24) of the norm in $\mathcal{W}_\gamma(\bar{U})$ we have

$$\|\psi_1 - \psi_2\| \leq L \|\varphi_1 - \varphi_2\|,$$

which means that $\rho(\psi_1, \psi_2) \leq L \rho(\varphi_1, \varphi_2)$, where $L < 1$ in view on (23).

By the Banach fixed point theorem, there is exactly one solution $\bar{\varphi} \in W_\gamma(\bar{U})$ of (1) satisfying the condition (12). This solution is given as the limit of series of successive approximations.

$$\varphi_n(x) = h(\varphi_{n-1}[f(x)]) + g(x), \quad n \in N, \quad x \in \bar{U}$$

where $\varphi_0 \in A_{r_0}$. This sequence converges in the sense of the norm of $W_\gamma(\bar{U})$. By Lemma 4 in [7], there exists the unique extension φ of $\bar{\varphi}$ to the whole interval I such that $\varphi = \bar{\varphi}$ for $x \in \bar{U}$ and φ satisfies the equation (1) in I . This completes the proof.

Conclusions

In this paper, applying the Banach contraction principle, a theorem on the existence and uniqueness of W_γ -solutions of nonlinear iterative functional equation (1) has been proved. The suitable unique solution is determined as a limit of sequence of successive approximations.

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