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THE NON-KELLER MAPPING WITH ONE ZERO AT INFINITY

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Abstract. In this paper the polynomial mapping of two complex variables having one zero at infinity is considered. Unlike with Keller mapping, if determinant of the Jacobian of this mapping is constant then it must be zero.

Keywords: *Jacobian, zero at infinity, polynomial mappings, Keller mappings*

1. Introduction

The main question of this paper concerns an attempt to analyze the problems connected with the special kind of polynomial mapping of two complex variables: non-Keller mapping. In [1, 2] the rare mappings with one zero at infinity was analyzed. It is shown that the Jacobian of non-Keller mapping being constant must vanish. It is a new mapping unlike with Keller maps that were studied over the last fifteen years [3-6].

In the generic case, the polynomial mappings with one zero at infinity are probably non-invertible. Namely, let

$$f = X^{p} + f_{p-1} + f_{p-2} + f_{p-3} + \ldots + f_1 \tag{1}$$

and

$$h = X^{q} + h_{q-1} + h_{q-2} + h_{q-3} + \ldots + h_1 \tag{2}$$

be coordinates of the polynomial mappings of two complex variables.
Now, let $p > q \geq 2$. It seems that in the generic case, if $\mathrm{Jac}(f, h) = \mathrm{const}$ then $\mathrm{Jac}(f, h) = 0$.

2. The non-invertible mapping

Let

$$f = X^k + f_{k-1} + f_{k-2} + f_{k-3} + ... + f_1 \tag{3}$$

and

$$h = X^2 + h_1 \tag{4}$$

where $k \geq 3$. The forms $f_i \in \mathbf{C}[X, Y]$ have degree $1 \leq i \leq k-1$.

Let additionally

$$\frac{k}{2} X^{k-2} h_1 + a_{k-1} X^{k-1} = f_{k-1} \tag{5}$$

where $a_{k-1} \neq 0$.

Lemma. Let

$$\mathrm{Jac}(f, h) = \mathrm{Jac}(f_1, h_1) = \mathrm{const} \tag{6}$$

Then

$$\mathrm{Jac}(f, h) = 0 \tag{7}$$

Proof. Let

$$\begin{aligned} f &= X^k + f_{k-1}\big|^{1)} + f_{k-2}\big|^{2)} + f_{k-3}\big|^{3)} + ... + f_1\big|^{k-1)} \\ h &= X^2 \ + \ h_1\big| + \quad 0\big| + \quad 0\big| \ + ... + \ 0\big| \end{aligned} \tag{8}$$

where $k \geq 3$.
In the next steps we have

$$1) \qquad \mathrm{Jac}(X^k, h_1) = \mathrm{Jac}(X^2, f_{k-1}) \tag{9}$$

so

$$kX^{k-1}\,\mathrm{Jac}(X, h_1) = 2X\,\mathrm{Jac}(X, f_{k-1}) \tag{10}$$

and

$$\frac{k}{2}X^{k-2}h_1 + a_{k-1}X^{k-1} = f_{k-1} \tag{11}$$

We assume further that $a_{k-1} \neq 0$

$$2) \qquad \mathrm{Jac}\left(f_{k-1}, h_1\right) = \mathrm{Jac}\left(X^2, f_{k-2}\right) \tag{12}$$

Substituting (11) into (12) we get

$$\begin{aligned}
\mathrm{Jac}\left(f_{k-1}, h_1\right) &= \mathrm{Jac}\left(\frac{k}{2}X^{k-2}h_1 + a_{k-1}X^{k-1},\, h_1\right) \\
&= \frac{k}{2}\mathrm{Jac}\left(X^{k-2}h_1, h_1\right) + \left(k-1\right)a_{k-1}X^{k-2}\ \mathrm{Jac}\left(X, h_1\right) \\
&= \frac{k}{2}h_1\,\mathrm{Jac}\left(X^{k-2}, h_1\right) + \left(k-1\right)a_{k-1}X^{k-2}\ \mathrm{Jac}\left(X, h_1\right) \\
&= \frac{k(k-2)}{2}X^{k-3}h_1\,\mathrm{Jac}\left(X, h_1\right) + \left(k-1\right)a_{k-1}X^{k-2}\ \mathrm{Jac}\left(X, h_1\right)
\end{aligned} \tag{13}$$

Therefore from (12) we obtain

$$\frac{k(k-2)}{2}X^{k-3}h_1\,\mathrm{Jac}\left(X, h_1\right) + \left(k-1\right)a_{k-1}X^{k-2}\ \mathrm{Jac}\left(X, h_1\right) = 2X\,\mathrm{Jac}\left(X, f_{k-2}\right) \tag{14}$$

Hence, dividing by $2X$, we have

$$\frac{k(k-2)}{2^2}X^{k-4}h_1\,\mathrm{Jac}\left(X, h_1\right) + \frac{\left(k-1\right)}{2}a_{k-1}X^{k-3}\ \mathrm{Jac}\left(X, h_1\right) = \mathrm{Jac}\left(X, f_{k-2}\right) \tag{15}$$

Subsequently

$$\frac{k(k-2)}{2^2\cdot 2!}X^{k-4}h_1^2 + \frac{\left(k-1\right)}{2}a_{k-1}X^{k-3}\ h_1 + a_{k-2}X^{k-2} = f_{k-2} \tag{16}$$

$$3) \qquad \mathrm{Jac}\left(f_{k-2}, h_1\right) = \mathrm{Jac}\left(X^2, f_{k-3}\right) \tag{17}$$

Substituting (16) into (17) we get

$$\begin{aligned}
\mathrm{Jac}\left(f_{k-2},h_1\right) &= \mathrm{Jac}\left(\frac{k(k-2)}{2^3}X^{k-4}\,h_1^2+\frac{k-1}{2}a_{k-1}\,X^{k-3}h_1+a_{k-2}X^{k-2},\,h_1\right)\\
&= \frac{k(k-2)}{2^3}\mathrm{Jac}\left(X^{k-4}\,h_1^2,h_1\right)+\frac{(k-1)}{2}a_{k-1}\mathrm{Jac}\left(X^{k-3}h_1,h_1\right)\\
&\quad +(k-2)a_{k-2}X^{k-3}\,\mathrm{Jac}\left(X,h_1\right)\\
&= \frac{k(k-2)}{2^3}h_1^2\,\mathrm{Jac}\left(X^{k-4},h_1\right)+\frac{(k-1)}{2}a_{k-1}h_1\mathrm{Jac}\left(X^{k-3},h_1\right)\\
&\quad +(k-2)a_{k-2}X^{k-3}\,\mathrm{Jac}\left(X,h_1\right)\\
&= \frac{k(k-2)(k-4)}{2^3}X^{k-5}h_1^2\,\mathrm{Jac}\left(X,h_1\right)\\
&\quad +\frac{(k-1)(k-3)}{2}a_{k-1}X^{k-4}h_1\mathrm{Jac}\left(X,h_1\right)\\
&\quad +(k-2)a_{k-2}X^{k-3}\,\mathrm{Jac}\left(X,h_1\right)
\end{aligned}\tag{18}$$

Therefore the formula (17) gives

$$\begin{aligned}
&\frac{k(k-2)(k-4)}{2^3}X^{k-5}h_1^2\,\mathrm{Jac}\left(X,h_1\right)+\frac{(k-1)(k-3)}{2}a_{k-1}X^{k-4}h_1\mathrm{Jac}\left(X,h_1\right)\\
&+(k-2)a_{k-2}X^{k-3}\,\mathrm{Jac}\left(X,h_1\right)=2X\,\mathrm{Jac}\left(X,f_{k-3}\right)
\end{aligned}\tag{19}$$

Similarly, dividing by $2X$, we have

$$\begin{aligned}
&\frac{k(k-2)(k-4)}{2^3 2}X^{k-6}h_1^2\,\mathrm{Jac}\left(X,h_1\right)+\frac{(k-1)(k-3)}{2^2}a_{k-1}X^{k-5}h_1\mathrm{Jac}\left(X,h_1\right)\\
&+\frac{(k-2)}{2}a_{k-2}X^{k-4}\,\mathrm{Jac}\left(X,h_1\right)=\mathrm{Jac}\left(X,f_{k-3}\right)
\end{aligned}\tag{20}$$

Consequently

$$\begin{aligned}
&\frac{k(k-2)(k-4)}{2^3 3!}X^{k-6}h_1^3+\frac{(k-1)(k-3)}{2^2 2!}a_{k-1}X^{k-5}h_1^2\\
&+\frac{(k-2)}{2}a_{k-2}X^{k-4}\,h_1+a_{k-3}X^{k-3}=f_{k-3}
\end{aligned}\tag{21}$$

etc.

Case 1. Suppose

$$k = 2l + 1, \quad l \geq 1 \tag{22}$$

In the step $l+1$ we obtain

$$\frac{(2l+1)\cdot\left(2l-1\right)\cdot\ldots\cdot 3\cdot 1}{2^l(l+1)!}h_1^{l+1} + X\left(\ldots\right) = 2X f_l \tag{23}$$

Therefore X divides h_1^{l+1}, so

$$h_1 = B_1 X \tag{24}$$

and

$$f_{2l} = A_{2l}X^{2l}, \ldots, f_1 = A_1 X \tag{25}$$

Subsequently

$$\mathrm{Jac}(f, h) = \mathrm{Jac}(f_1, h_1) = 0 \tag{26}$$

Case 2. Suppose

$$k = 2l + 2, \quad l \geq 1 \tag{27}$$

Then

$$\frac{2l+2}{2}X^{2l}h_1 + a_{2l+1}X^{2l+1} = f_{2l+1} \tag{28}$$

where $a_{2l+1} \neq 0$.
In the step $l+2$ we obtain

$$\frac{(2l+1)\cdot\left(2l-1\right)\cdot\ldots\cdot 3\cdot 1}{2^l(l+1)!}a_{2l+1}h_1^{l+1} + X(\ldots) = 2X f_l \tag{29}$$

Therefore X divides h_1^{l+1}, so

$$h_1 = \tilde{B}_1 X \tag{30}$$

and

$$f_{2l+1} = \tilde{A}_{2l+1}X^{2l+1}, \ldots, f_1 = \tilde{A}_1 X \tag{31}$$

Subsequently

$$\operatorname{Jac}(f,h) = \operatorname{Jac}(f_1,h_1) = 0 \tag{32}$$

This completes the proof.

3. Conclusion

In this paper the particular case of the form h for $q = 2$ was investigated. It is not hard to show that if $q = 3$ then $h = X^3 + B_2X^2 + h_1$. Consequently, in a similar way, we can prove that if the determinant of non-Keller mapping is constant then in generic case, it must vanish.

It seems that in the case $q \geq 3$ the second coordinates h of considering mapping has the form $h = X^q + B_{q-1}X^{q-1} + \ldots + B_2X^2 + h_1$.

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