O-SPECIES AND TENSOR ALGEBRAS

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Abstract. In this paper we consider O-species and their representations. These O-species are a type of a generalization of a species introduced by Gabriel. We also consider the tensor algebras of such O-species. It is proved that the category of all representations of an O-species and the category of all right modules over the corresponding tensor algebra are naturally equivalent.

Keywords: species, O-species, representations of O-species, tensor algebra, O-species of bounded representation type, diagram of O-species

1. Introduction

In this paper we consider O-species, which generalize the notion of species introduced by Gabriel in [1]. Recall this definition:

Definition 1.1. (Gabriel [1]). Let $I$ be a finite index set. A species $L = (F_i, M_{ij})_{i,j \in I}$ is a finite family $(F_i)_{i \in I}$ of division rings together with a family $(M_{ij})_{i,j \in I}$ of $(F_i, F_j)$-bimodules.

We say that $(F_i, M_{ij})_{i,j \in I}$ is a $K$-species if all $F_i$ are finite dimensional and central over the common commutative subfield $K$ which acts centrally on $M_{ij}$, i.e. $\lambda m = m \lambda$ for all $\lambda \in K$ and all $m \in M_{ij}$. We also assume that each bimodule $M_{ij}$ is a finite dimensional vector space over $K$. K-species is a $K$-quiver if $F_i = K$ for each $i$.

Definition 1.2. A representation $(V, \varphi)$ of a species $L = (F_i, M_{ij})_{i,j \in I}$ (or an $L$-representation) is a family of right $F_i$-modules $V_i$ and $F_i$-linear mappings:

$$ j \varphi : V_i \otimes_{F_i} M_{ij} \to V_j $$

for each $i, j \in I$. Such a representation is called finite dimensional, provided all the spaces $V_i$ are finite dimensional vector spaces.
Let \( V = (V_i, \phi_i) \) and \( W = (W_i, \psi_i) \) be two \( L \)-representations. An \( L \)-morphism \( \Psi: V \to W \) is a set of \( F_i \)-linear maps \( \alpha: V_i \to W_i \) such that

\[
_j \psi_i (\alpha_i \otimes 1) = \alpha_{j,j} \phi_i \tag{1.4}
\]

Two representations \( (V_i, \phi_i) \) and \( W = (W_i, \psi_i) \) are called **equivalent** if there is a set of isomorphisms \( \alpha_i \) from the \( F_i \)-module \( V_i \) to the \( F_i \)-module \( W_i \) such that the (1.4) holds for all \( i,j \in I \).

A representation \( (V_i, \phi_i) \) is called **indecomposable**, if there are no non-zero sets of subspaces \( (U_i) \) and \( (W_i) \) such that \( V_i = U_i \oplus W_i \) and \( j \phi_i = j \psi_i \oplus j \tau_i \), where

\[
j \psi_i: U_i \otimes_{F_i} M_j \to U_j \tag{1.5}
\]

\[
j \tau_i: W_i \otimes_{F_i} M_j \to W_j \tag{1.6}
\]

One defines the direct sum of two \( L \)-representations in the obvious way.

Denote by \( \text{Rep}(L) \) the category of all \( L \)-representations, and by \( \text{rep}(L) \) the category of finite dimensional \( L \)-representations, whose objects are \( L \)-representations and whose morphisms are as defined above.

**Definition 1.7.** [2] A species \( L = (F_i, M_j)_{i,j \in I} \) is said to be of **finite type**, if the number of indecomposable non-isomorphic finite dimensional representations is finite.

A species \( L = (F_i, M_j)_{i,j \in I} \) is said to be of **strongly unbounded type** if it possesses the following three properties:

1. \( L \) has indecomposable objects of arbitrary large finite dimension.
2. If \( L \) contains a finite dimensional object with an infinite endomorphism ring, then there is an infinite number of (finite) dimensions \( d \) such that, for each \( d \), the species \( L \) has infinitely many (non-isomorphic) indecomposable objects of dimension \( d \).
3. \( L \) has indecomposable objects of infinite dimension.

Dlab and Ringel proved in [2, Theorem E] that any \( K \)-species is either of finite or of strongly unbounded type.

With any species \( L = (F_i, M_j)_{i,j \in I} \) one can define the tensor algebra in the following way. Let \( B = \prod_{i \in I} F_i \), and let \( M = \bigoplus_{i,j \in I} M_j \). Then \( B \) is a ring and \( M \) naturally becomes a \((B, B)\)-bimodule. The **tensor algebra** of the \((B, B)\)-bimodule \( M \) is the graded ring
with component-wise addition and the multiplication induced by taking tensor products.

If \( L \) is a \( K \)-species, then \( T(L) \) is a finite dimensional \( K \)-algebra.

**Theorem 1.9.** (Dlab, Ringel [2, Proposition 10.1]). Let \( L \) be a \( K \)-species. Then the category \( \text{Rep}(L) \) of all representations of \( L \) and the category \( \text{Mod}_r(T(L)) \) of all right \( T(L) \)-modules are equivalent.

### 2. \( O \)-species and their representations

In this section we consider the notion of \( O \)-species, which generalizes the notion of species considered in [1].

Let \( \{O_i\} \) be a family of discrete valuation rings (not necessarily commutative) \( O_i \) with radicals \( R_i \) and skew fields of fractions \( D_i \), for \( i = 1, 2, \ldots, k \), and let \( \{D_j\} \), for \( j = k + 1, \ldots, n \), be a family of skew fields. Let \((n_1, n_2, \ldots, n_k)\) be a set of natural numbers. Write

\[
\begin{bmatrix}
O_1 & O_1 & \cdots & O_1 \\
R_1 & O_1 & \cdots & O_1 \\
\vdots & \vdots & \ddots & \vdots \\
R_1 & R_1 & \cdots & O_1
\end{bmatrix},
\]

which is a subring in the matrix ring \( M_{n_i}(D_i) \). It is easy to see that each \( H_{n_i}(O_i) \) is a Noetherian serial prime hereditary ring. Write \( F_i = H_{n_i}(O_i) \) for \( i = 1, 2, \ldots, k \), and \( F_j = D_j \) for \( j = k + 1, \ldots, n \). Then, by the Goldie theorem, there exists a classical ring of fractions \( \tilde{F}_i \) for \( i = 1, 2, \ldots, n \).

Consider the following generalization of a species.

**Definition 2.1.** An \( O \)-species is a set \( \Omega = (F_i, M_i)_{i,j \in I} \), where \( F_i = H_{n_i}(O_i) \) for \( i = 1, 2, \ldots, k \), and \( F_j = D_j \) for \( j = k + 1, \ldots, n \), and moreover \( M_i \) is an \( (\tilde{F}_i, \tilde{F}_j) \)-bimodule, which is finite dimensional as a right \( D_j \)-vector space and as a left \( D_i \)-vector space.

An \( O \)-species \( \Omega \) is called a \( (D, O) \)-species if all \( O \) have a common skew field of fractions \( D \), i.e. all \( D_i \) are equal to a fixed skew field \( D \) and
for some natural number $n_i (i = 1, 2, ..., n)$.

An $O$-species $\Omega$ is called a $(K, O)$-species, if all $D_i (i = 1, 2, ..., n)$ contain a common central subfield $K$ of finite index in such a way that $\lambda m = m \lambda$ for all $\lambda \in K$ and all $m \in M_i$ (moreover, each bimodule $M_i$ is a finite dimensional vector space over $K$). It is a $(K, O)$-quiver if moreover $D_i = D$ for each $i$.

Everywhere in this paper we will consider $O$-species without oriented cycles and loops, i.e. we will assume that $M_i = 0$, and if $M_i \neq 0$, then $M_i = 0$. A vertex $i$ is said to be marked if $F_i = H_{n_i} (O_i)$.

We will also assume that all marked vertices are minimal, i.e. $M_i = 0$ if $F_i = H_{n_i} (O_i)$, and that $M_j = 0$ if $i, j$ are marked vertices.

**Definition 2.3.** The *diagram* of an $O$-species $\Omega = \{F_i, M_{ij}\}_{i,j \in I}$ is defined in the following way:

1. The set of vertices is a finite set $I = \{1, 2, ..., n\}$.
2. The finite subset $I_0 = \{1, 2, ..., k\}$ of $I$ is a set of marked points.
3. The vertex $i$ connects with the vertex $j$ by $t_{ij}$ arrows, where

$$t_{ij} = \frac{1}{n_i} \dim_D (M_j) \times \dim_i (M_j),$$

moreover, we assume that $n_i = 1$ if $F_i = D_i$.

Similar to species we can define representations of $O$-species in the following way.

**Definition 2.4.** A representation $(M_i, V_{ir}, \phi_{ir}, \psi_{ir})$ of an $O$-species $\Omega = \{F_i, M_{ij}\}_{i,j \in I}$ is a family of right $F_i$-modules $M_i (i = 1, 2, ..., n)$ and $D_j$-linear maps:

$$\phi_{ir} : M_j \otimes_{D_j} M_i \rightarrow V_j$$

for each $i = 1, 2, ..., n$; $j = 1, 2, ..., n$; and

$$\psi_{ir} : V_r \otimes_{D_r} M_j \rightarrow V_j$$

for each $r = 1, 2, ..., n$.

**Definition 2.5.** Two representations $M = (M_i, V_{ir}, \phi_{ir}, \psi_{ir})$ and $M' = (M'_i, V'_{ir}, \phi'_{ir}, \psi'_{ir})$ are called equivalent if there is a set of isomorphisms $\alpha_i$ of $F_i$-modules from $M_i$ to
$M'_r$ and a set of isomorphisms $\beta_r$ of $D_r$-vector spaces from $V_r$ to $V'_r$ such that for each $i = 1, 2, ..., k; r, j = k + 1, k + 2, ..., n$ the following equalities hold:

$$j\varphi'_r(\alpha \otimes 1) = \beta_{j', j} \varphi_i$$

(2.6)

$$j\psi'_r(\beta_r \otimes 1) = \beta_{j', j} \psi_r$$

(2.7)

In a natural way one can define the notions of a direct sum of representations and of an indecomposable representation.

The set of all representations of an $O$-species $\Omega = (F_i, M_j)_{i,j}$ can be turned into a category $\mathcal{R}(\Omega)$, whose objects are representations $M = (M_i, V_r, \varphi_i, \psi_r)$, and a morphism from object $M = (M_i, V_r, \varphi_i, \psi_r)$ to object $M' = (M'_i, V'_r, \varphi'_i, \psi'_r)$ is a set of homomorphisms $\alpha_i$ of $H_{n_i}(O_i)$-modules $M_i$ to $M'_i$, and a set of homomorphisms $\beta_r$ of $D_r$-vector spaces from $V_r$ to $V'_r$ such that for each $i = 1, 2, ..., k; r, j = k + 1, k + 2, ..., n$ the equalities (2.6) and (2.7) hold.

3. Tensor algebra of $O$-species

For any $O$-species $\Omega = (F_i, M_j)_{i,j}$ one can construct a tensor algebra of bimodules $T(\Omega)$. Let $A = \bigoplus_{i=1}^q F_i$, $B = \bigoplus_{i,j} M_{ij}$. Then $B$ is an $(A, A)$-bimodule and we can define a tensor algebra $T_A(B)$ of the bimodule $B$ over the ring $A$ in the following way:

$$T_A(B) = A \oplus B \oplus B^2 \oplus ... \oplus B^n \oplus ...$$

(3.1)

is a graded ring, where $B^n = B \otimes_A B^{n-1}$ for $n > 1$, and multiplication in $T_A(B)$ is given by the natural $A$-bilinear map:

$$B^n \times B^m \to B^n \otimes_A B^m = B^{n+m}$$

(3.2)

Then $T(\Omega) = T_A(B)$ is the tensor algebra corresponding to an $O$-species $\Omega$.

**Proposition 3.3.** Let $\Omega$ be an $O$-species. Then the category $\mathcal{R}(\Omega)$ of all representations of $\Omega$ and the category $\text{Mod}_T(\Omega)$ of all right $T(\Omega)$-modules are naturally equivalent.

**Proof.** Form two functors $R: \text{Mod}_T(\Omega) \to \mathcal{R}(\Omega)$ and $P: \mathcal{R}(\Omega) \to \text{Mod}_T(\Omega)$ in the following way. Let $X_T(\Omega)$ be a right $T(\Omega)$-module. Since $A$ is a subring in $T(\Omega)$, $X$ can be considered as a right $A$-module. Then
where $M_i$ is an $H_{n_j}(O_i)$-module, and $V_r$ is a $D_r$-vector space; moreover, $M_iH_{n_j}(O_j) = 0$ for $i \neq j$, and $V_iD_j = 0$ for $r \neq s$. Since $B$ is an $(A, A)$-bimodule, one can define an $A$-homomorphism $\varphi : X \otimes_A B \to X_A$. Taking into account that $M_i \otimes_A M_j = 0$ for $i \neq s$, the map $\varphi$ is defined in the following way:

$$\varphi : \left( \bigoplus_{i=1}^{k} (M_i \otimes_A M_j) \right) \otimes \left( \bigoplus_{r=k+1}^{n} (V_r \otimes_A M_j) \right) \to \bigoplus_{r=k+1}^{n} V_r$$

(3.5)

Since $M_i \otimes_A M_j$ is mapping into $V_j$, and $V_r \otimes_A M_j$ is mapping into $V_j$, $\varphi$ defines a set of $D_r$-homomorphisms:

$$\varphi_i : M_i \otimes_A M_j = M_i \otimes H_{n_j}(O_j) M_j \to V_j$$

(3.6)

$$\varphi_r : V_r \otimes_A M_j = V_r \otimes D_r M_j \to V_j$$

(3.7)

for $i = 1, 2, ..., k; r, j = k + 1, ..., n$.

Now one can define $R(X_{\Omega}) = (M_i, V_r, \varphi_i, \psi_r)$. Let $X, Y$ be two right $T(\Omega)$-modules, let $\alpha : X \to Y$ be a homomorphism, and let $R(X) = (M_i, V_r, \varphi_i, \psi_r), R(Y) = (N_i, W_r, \bar{\varphi}_i, \bar{\psi}_r)$. Let's define a morphism from $R(X)$ to $R(Y)$. Since $\alpha$ is an $A$-homomorphism, $\alpha(M_i) \subseteq N_i$, $\alpha(V_r) \subseteq W_r$, i.e., $\alpha$ defines a family of $H_{n_i}(O_i)$-homomorphisms $\alpha_i : M_i \to N_i$ and a family of $D_r$-homomorphisms $\beta_r : V_r \to W_r$, which are the restrictions of $\alpha$ to $M_i$ and $V_r$. Therefore one can set $R(\alpha) = \{(\alpha_i), (\beta_r)\}$. Since $\alpha$ is a $T(\Omega)$-homomorphism,

$$\varphi_i (\alpha_i \otimes 1) = \alpha_{i,j} \varphi_i$$

(3.8)

and

$$\varphi_r (\beta_r \otimes 1) = \beta_{j,r} \psi_r$$

(3.9)

for $i = 1, 2, ..., k; r, j = k + 1, ..., n$. Therefore $R(\alpha)$ is a morphism in the category $R(\Omega)$.

Conversely, let $\Omega = (F_n, \rho)$ and there is given a representation $M = (M_i, V_r, \varphi_i, \psi_r)$. Then one can define $P(M)$ in the following way:

$$P(M) = X = \left( \bigoplus_{i=1}^{k} M_i \right) \oplus \left( \bigoplus_{r=k+1}^{n} V_r \right)$$

(3.10)
We define an action of
\[ A = \left( \bigoplus_{i=1}^{k} H_{n_i}(O_i) \right) \bigoplus \left( \bigoplus_{r=k+1}^{n} D_r \right) \] (3.11)
on \( M_i \) by means of the projection \( A \to H_{n_i}(O_i) \) and an action of \( A \) on \( V_r \) by means of the projection \( A \to D_r \). We define an action of \( B^n \) on \( X \) by induction of \( \phi^{(n)} : X \otimes_A B^n \to X \) as follows:
\[ \phi^{(1)} = \bigoplus_{i,j} \phi_{i,j} \otimes \psi_{r} : X \otimes_A B = \left( \bigoplus_{i=1}^{k} (M_i \otimes_A M_j) \right) \bigoplus \left( \bigoplus_{r=k+1}^{n} (V_r \otimes_A M_j) \right) = \left( \bigoplus_{i=1}^{k} (M_i \otimes_{H_{n_i}(O_i)} M_j) \right) \bigoplus \left( \bigoplus_{r=k+1}^{n} (V_r \otimes_{D_r} M_j) \right) \to \bigoplus_{r=k+1}^{n} V_r \subseteq X. \]
\[ \phi^{(n+1)} = \phi(\phi^{(n)} \otimes 1) : X \otimes_A B^{(n+1)} = (X \otimes_A B) \otimes_A B^n \xrightarrow{\phi^{(n)} \otimes 1} X \otimes_A B \xrightarrow{\phi} X \]
If \( \alpha = \{ \{ \alpha_i \}, \{ \beta_r \} \} \) is a morphism of a representation \( M = (M_i, V_r, \psi_{ \alpha_i}, \psi_{ \beta_r}) \) to a representation \( M' = (M_i', V_r', \psi_{ \alpha_i}', \psi_{ \beta_r}') \), then
\[ \varphi = \bigoplus_i \alpha_i \bigoplus_r \beta_r : X = \bigoplus_i M_i \bigoplus_r V_r \to \bigoplus_i M_i' \bigoplus_r V_r' \] (3.12)
is a \( T(\Omega) \)-homomorphism and therefore \( P(\alpha) = \varphi \).

It is not difficult to show that \( R, P \) are mutually inverse functors and they give an equivalence of categories \( \text{Mod} \, T(\Omega) \) and \( \mathcal{R}(\Omega) \).

Recall that an Artinian ring \( A \) is of finite representation type if \( A \) has only a finite number of indecomposable finitely generated right \( A \)-modules up to isomorphism.

A ring \( A \) is of (right) bounded representation type (see [3, 4]) if there is an upper bound on the number of generators required for indecomposable finitely presented right \( A \)-modules.

Denote by \( \mu(M_i) \) the minimal number of generators of an \( H_{n_i}(O_i) \)-module \( M_i \), and denote by \( d_r = \dim_{D_r}(V_r) \) the dimension of vector space \( V_r \) over \( D_r \). The dimension of a representation \( M = (M_i, V_r, \psi_{ \alpha_i}, \psi_{ \beta_r}) \) is the number
\[ d = \dim M = \sum_{i=1}^{n} \mu(M_i) + \sum_{r=k+1}^{n} d_r \] (3.13)
Definition 3.14. An $O$-species $\Omega$ is said to be of \textbf{bounded representation type} if the dimensions of its indecomposable finite dimensional representations have an upper bound.

Corollary 3.15. An $O$-species $\Omega$ is of bounded representation type if and only if the tensor algebra $T(\Omega)$ is of bounded representation type.

\textit{Proof.} If $\Omega$ is an $O$-species of bounded representation type, then there exists $N > 0$ such that $\dim M < N$ for any indecomposable finite dimensional representation $M$. Then for any finitely generated $T(\Omega)$-module $X$ we have $\mu(X) < N_1$, where $N_1$ is some fixed number depending on $N$, i.e. $T(\Omega)$ is a ring of bounded representation type. The converse also holds: if $T(\Omega)$ is a ring of bounded representation type, then $\Omega$ is an $O$-species of bounded representation type.

Corollary 3.16. Let $\Omega_1$ be a $D$-species, which is a subspecies of a $(D, O)$-species $\Omega$. If $\Omega$ is of bounded representation type, then $\Omega_1$ is of finite type.

\textit{Proof.} Since $\Omega$ is of bounded representation type, each of its subspecies is of bounded representation type as well. So $\Omega_1$ is of bounded representation type, and, by corollary 3.15, its tensor algebra is of bounded representation type, as well. Since $\Omega_1$ is a $D$-species, its tensor algebra is an Artinian ring. So it is of finite representation type, by [5]. Therefore, $\Omega_1$ is also of finite representation type.

3. Conclusion

In this paper we introduced $O$-species and the tensor algebras corresponding to them. These $O$-species are some generalizations of species first introduced by Gabriel in [1]. We consider the notion of a representation of an $O$-species. In this paper we prove that the category of all representations of $O$-species $\Omega$ and the category of all right modules over a tensor algebra $T(\Omega)$ are naturally equivalent.

References