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ON A CERTAIN PROPERTY OF GENERALIZED HÖLDER FUNCTIONS

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Abstract. In this paper some properties of functions belonging to the space $W_\gamma[a,b]$ of generalized Hölder functions are considered. These functions are r-times differentiable and their r-th derivatives satisfy the generalized Hölder condition. The main result of the paper is a proof of the fundamental lemma that the recursive model-defined functions $h_k: I \times R^{k+1} \to R,\ k = 0,1,\dots,r$ are a special form and belong to the space $W_\gamma[a,b]$.

Keywords: *Lipschitz condition, generalized Hölder condition, γ-Hölder condition*

1. Introduction

In the paper [1] we introduced a function space $W_\gamma[a,b]$ and proved some of its properties. In the books [2] by Kuczma and [3] by Kuczma, Choczewski and Ger the existence and uniqueness of the solution of a certain functional equation in various function spaces (such as $Lip[a,b], C^r[a,b], BV[a,b]$) were proved. A similar result for the linear and nonlinear functional equation in the $W_\gamma[a,b]$-space was obtained in [4, 5]. In our paper we prove a fundamental lemma describing the form of the functions in $W_\gamma[a,b]$. This lemma can be applied in proof of theorem concerning the existence and uniqueness of solutions of a functional equation. Examples of such applications of the introduced lemma will be presented in our next paper.

2. Main result

We recall the definition of the space $W_\gamma[a,b]$.

Let $[a,b]$ be a closed interval, where $a,b \in R,\ a < b,\ d := b - a$. We assume that the following condition is fulfilled:

$$(\Gamma)\ \gamma: [0,d] \to [0,\infty) \text{ is increasing and concave, } \gamma(0) = 0,$$
$$\lim_{t\to 0^+} \gamma(t) = \gamma(0),\ \lim_{t\to d^-} \gamma(t) = \gamma(d).$$

Definition 1. Given $r \in N$, denote by $W_\gamma[a,b]$ the set of all r-times differentiable functions defined on the interval $[a,b]$ with values in R, such that their r-th derivatives satisfy the following condition: there exists a constant $M \geq 0$ such that

$$\left|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})\right| \leq M\gamma(|x - \bar{x}|),\ \ \bar{x}, x \in [a,b] \tag{1}$$

where γ fulfills condition (Γ).

It is easily seen that $W_\gamma[a,b]$ contains the class of all r-times differentiable functions $\varphi\colon [a,b] \to R$, whose r-th derivatives satisfy the Lipschitz condition on $[a,b]$. This class is denoted by $LipC^r[a,b]$. Thus we have

$$LipC^r[a,b] \subset W_\gamma[a,b].$$

Denote by $\gamma'_+(0)$ the right derivative of γ at $t = 0$. By (Γ) we have $0 \leq \gamma'_+(0) \leq +\infty$.
For $\varphi \in W_\gamma[a,b]$ and by the condition (Γ) we obtain

$$\left|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})\right| \leq M\gamma(|x - \bar{x}|) \leq M\gamma'_+(0)|x - \bar{x}|,\ \ \bar{x}, x \in [a,b]$$

i.e. $\varphi^{(r)}$ fulfills an ordinary *Lipschitz condition* with the constant $K = M\gamma'_+(0)$.
Thus if $\varphi \in W_\gamma[a,b]$ and $\gamma'_+(0)$ is finite, then $\varphi \in LipC^r[a,b]$. Thus we get $LipC^r[a,b] = W_\gamma[a,b]$.
Therefore only the case $\gamma'_+(0) = +\infty$ is of interest.

The functions of the form $\gamma(t) = t^\alpha$, where $0 < \alpha < 1$, $t \in [0,d]$, fulfill the assumption (Γ) and moreover $\gamma'_+(0) = +\infty$. Therefore the condition (1) is called *the generalized Hölder condition* or the *γ-Hölder condition.*

The space $W_\gamma[a,b]$ with the norm

$$\|\varphi\| \coloneqq \sum_{k=0}^{r}\left|\varphi^{(k)}(a)\right| + sup\left\{\frac{\left|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})\right|}{\gamma(|x-\bar{x}|)};\ x, \bar{x} \epsilon [a,b], x \neq \bar{x}\right\}$$

is a real normed vector space. Moreover, it is a Banach space.

Consider the functional equation

$$\varphi(x) = h(\varphi[f(x)]) + g(x)$$

We assume that the given functions fulfill the following conditions:

(i) $f\colon I \to I,\ f\epsilon W_\gamma(I),\ \sup_I|f'| \leq 1$

(ii) $g\colon I \to R,\ g\epsilon W_\gamma(I)$.

(iii) $h\colon R \to R, h$ is of the class C^r and $h^{(r)}$ fulfills the Lipschitz condition in R.

We define functions $h_k\colon I \times R^{k+1} \to R, k = 0,1,\dots,r$ by the formula

$$\begin{cases} h_0(x,y_0) := h(y_0) + g(x) \\ h_{k+1}(x,y_0,\dots,y_{k+1}) := \frac{\partial h_k}{\partial x} + f'(x)\left(\frac{\partial h_k}{\partial y_0}y_1 + \dots + \frac{\partial h_k}{\partial y_k}y_{k+1}\right) \end{cases} \quad (2)$$

for $k = 0,1,\dots,r-1$.

Lemma 1. If the assumptions (i)-(iii) are fulfilled, then the functions h_k defined by (2) are of the form:

1. for $r = 1$

$$h_1(x,y_0,y_1) = h'(y_0)y_1f'(x) + g'(x); \quad (3)$$

2. for $r \geq 2,\ k = 2,\dots,r$

$$h_k(x,y_0,\dots,y_k) = p_k(x,y_0,\dots,y_{k-1}) + h'(y_0)y_k\big(f'(x)\big)^k + \\ + h'(y_0)y_1f^{(k)}(x) + g^{(k)}(x) \quad (4)$$

where

$$p_k(x,y_0,\dots,y_{k-1}) + h'(y_0)y_k\big(f'(x)\big)^k = \\ = \sum_{i=1}^{k} h^{(k-i+1)}(y_0)\sum_{\alpha_1+\dots+\alpha_i=k-i+1} u_{\alpha_1\dots\alpha_i,k}(x){y_1}^{\alpha_1}\dots{y_i}^{\alpha_i} \quad (5)$$

and the functions $u_{\alpha_1\dots\alpha_i,k}$ are of the class C^{r-k+1} in *I*, for all possible natural numbers $\alpha_1,\dots,\alpha_i$ such that $\alpha_1 + \dots + \alpha_i = k - i + 1, k = 2,\dots,r,\ i = 1,\dots,k$, (some of these functions are identically equal to zero).

Proof: The first part of thesis follows from the definition (2). We prove the second part by mathematical induction. For $k = 2$ by (2) we get

$$h_2(x,y_0,y_1,y_2) = h'(y_0)y_1f''(x) + h''(y_0)y_1^2\big(f'(x)\big)^2 + \\ + h'(y_0)y_2\big(f'(x)\big)^2 + g''(x).$$

Putting $p_2(x,y_0,y_1) = h''(y_0)y_1^2\big(f'(x)\big)^2$ we get the formula (5) for $k = 2$.

Indeed

$$p_2(x, y_0, y_1) + h'(y_0)y_2\big(f'(x)\big)^2 = h''(y_0)u_{2,2}(x)y_1^2 + h'(y_0)\big(u_{10,2}(x)y_1 + u_{01,2}(x)y_2\big)$$

where $u_{2,2}(x) = \big(f'(x)\big)^2$, $u_{10,2}(x) = 0$, $u_{01,2}(x) = \big(f'(x)\big)^2$.

Therefore $u_{\alpha_1 \dots \alpha_i,2} \epsilon C^{r-1}, \alpha_1 + \dots + \alpha_i = 2 - i + 1, i = 1,2$.

Suppose that for $l, 2 \leq l < r$ the formula (4) is true and that there exist the functions $u_{\alpha_1 \dots \alpha_i,l} \epsilon C^{r-l+1}$, $\alpha_1 + \dots + \alpha_i = l - i + 1, i = 1, \dots, l$, which satisfy the condition (5). We prove that in such a case the equations (4) and (5) also hold for $l + 1$. By the definition (2) we get

$$h_{l+1}(x, y_0, \dots, y_{l+1}) = f^{(l+1)}(x)h'(y_0)y_1 + g^{(l+1)}(x) +$$
$$+ \sum_{i=1}^{l} h^{(l-i+1)}(y_0) \sum_{\alpha_1+\dots+\alpha_i=l-i+1} u'_{\alpha_1\dots\alpha_i,l}(x){y_1}^{\alpha_1} \dots {y_i}^{\alpha_i} +$$
$$+ \sum_{i=1}^{l} h^{(l-i+2)}(y_0)f'(x)y_1 \sum_{\alpha_1+\dots+\alpha_i=l-i+1} u_{\alpha_1\dots\alpha_i,l}(x){y_1}^{\alpha_1} \dots {y_i}^{\alpha_i} +$$
$$+ f^{(l)}(x)h''(y_0)y_1^2 f'(x) +$$
$$+ \sum_{i=1}^{l} h^{(l-i+1)}(y_0) \sum_{\alpha_1+\dots+\alpha_i=l-i+1} u_{\alpha_1\dots\alpha_i,l}(x)f'(x) \cdot$$
$$\cdot \sum_{k=1}^{i} \alpha_k {y_1}^{\alpha_1} \dots {y_{k-1}}^{\alpha_{k-1}} {y_k}^{\alpha_k - 1} y_{k+1} {y_{k+1}}^{\alpha_{k+1}} \dots {y_i}^{\alpha_i} + f^{(l)}(x)h'(y_0)y_2 f'(x) =$$
$$= h^{(l+1)}(y_0) \sum_{\alpha_1=l} u_{\alpha_1,l}(x)f'(x)y_1 y_1^l + h^{(l)}(y_0)\big(\sum_{\alpha_1=l} u'_{\alpha_1,l}(x)y_1^l +$$
$$+ \sum_{\alpha_1+\alpha_2=l-1} u_{\alpha_1,\alpha_2,l}(x)f'(x)y_1 {y_1}^{\alpha_1}{y_2}^{\alpha_2} + \sum_{\alpha_1=l} u_{\alpha_1,l}(x)f'(x) l y_1^{l-1} y_2\big) +$$
$$+ \dots + h^{(l-i+2)}(y_0)\big(\sum_{\alpha_1+\dots+\alpha_{i-1}=l-i+2} u'_{\alpha_1\dots\alpha_{i-1},l}(x){y_1}^{\alpha_1} \dots {y_{i-1}}^{\alpha_{i-1}} +$$
$$+ \sum_{\alpha_1+\dots+\alpha_i=l-i+1} u_{\alpha_1\dots\alpha_i,l}(x)f'(x)y_1 {y_1}^{\alpha_1} \dots {y_i}^{\alpha_i} +$$
$$+ \sum_{\alpha_1+\dots+\alpha_{i-1}=l-i+2} u_{\alpha_1\dots\alpha_{i-1},l}(x)f'(x) \sum_{k=1}^{i-1} \alpha_k {y_1}^{\alpha_1} \dots {y_k}^{\alpha_k-1} {y_{k+1}}^{\alpha_{k+1}+1} \dots {y_{i-1}}^{\alpha_{i-1}}\big) +$$
$$+ \dots + h''(y_0)\big(\sum_{\alpha_1+\dots+\alpha_{l-1}=2} u'_{\alpha_1\dots\alpha_{l-1},l}(x){y_1}^{\alpha_1} \dots {y_{l-1}}^{\alpha_{l-1}} + f^{(l)}(x)f'(x)y_1^2 +$$
$$+ \sum_{\alpha_1+\dots+\alpha_l=1} u_{\alpha_1\dots\alpha_l,l}(x)f'(x)y_1 {y_1}^{\alpha_1} \dots {y_l}^{\alpha_l} +$$
$$+ \sum_{\alpha_1+\dots+\alpha_{l-1}=2} u_{\alpha_1\dots\alpha_{l-1},l}(x)f'(x) \sum_{k=1}^{l-1} \alpha_k {y_1}^{\alpha_1} \dots {y_k}^{\alpha_k-1} {y_{k+1}}^{\alpha_{k+1}+1} \dots {y_{l-1}}^{\alpha_{l-1}}\big) +$$
$$+ h'(y_0)\big(f^{(l)}(x)f'(x)y_2 + \sum_{\alpha_1+\dots+\alpha_l=1} u'_{\alpha_1\dots\alpha_l,l}(x){y_1}^{\alpha_1} \dots {y_l}^{\alpha_l} +$$
$$+ \sum_{\alpha_1+\dots+\alpha_l=1} u_{\alpha_1\dots\alpha_l,l}(x)f'(x) \sum_{k=1}^{l} \alpha_k {y_1}^{\alpha_1} \dots {y_k}^{\alpha_k-1} {y_{k+1}}^{\alpha_{k+1}+1} \dots {y_l}^{\alpha_l}\big).$$

We note that the coefficient of the expression $h^{(l+1)}$is the $(l+1)$-degree monomial of one variable y_1 multiplied by the function of the variable x. By the induction hypothesis $u_{\alpha_1,l} \in C^{r-l+1}, \alpha_1 = l$ taking $u_{\beta_1,l+1}(x) := u_{\alpha_1,l}(x)f'(x)$, $\beta_1 = l+1, \alpha_1 = l$, we get that $u_{\beta_1,l+1} \in C^{r-l+1}, l = 2, \dots, r, x \in I$.

The coefficient of the expression $h^{(l)}$ is the sum of l-degree monomials of the variables y_1, y_2. Due to the assumption, the coefficients of the monomials are at least of the class C^{r-1} due to the variable $x \in I$. Thus, the expression at $h^{(l)}$ can be written in the form

$$\sum_{\beta_1+\beta_2=1} u_{\beta_1,\beta_2,l+1}(x){y_1}^{\beta_1}{y_2}^{\beta_2},$$

where $u_{\beta_1,\beta_2,l+1} \in C^{r-1}$ (some of these functions are identically equal to zero).

Generalizing, each derivative in the form of $h^{(l-i+2)}$, $i = 1, \dots, l+1$ is multiplied by the sum of $(l-i+2)$-degree monomials of the variables $y_1, \dots, y_i$, where $i = 1, \dots, l+1$. The coefficients of the monomials are functions of the variable x, at least of the class C^{r-l} in I. Denote these functions by $u_{\beta_1 \dots \beta_i,l+1}$ for all possible numbers $\beta_1, \dots, \beta_i$ such that $\beta_1 + \dots + \beta_i = l-i+2, i = 1, \dots, l+1$ (some of these functions are identically equal to zero). Thus

$$h_{l+1}(x, y_0, \dots, y_{l+1}) =$$
$$= \sum_{i=1}^{l+1} h^{(l-i+2)}(y_0) \sum_{\beta_1+\dots+\beta_i= l-i+2} u_{\beta_1 \dots \beta_i,l+1}(x){y_1}^{\beta_1} \dots {y_i}^{\beta_i} +$$

$$+h'(y_0)y_1 f^{(l+1)}(x) + g^{(l+1)}(x),$$

where the functions $u_{\beta_1 \dots \beta_i,l+1} \in C^{r-l}, i = 1, \dots, l+1, l = 2, \dots, r-1$.

Therefore the equalities (4) and (5) are true for $k = 2, \dots, r$. This completes the proof.

Remark 1. If the assumptions (i)-(iii) are fulfilled, then the functions $h_r: I \times R^{k+1} \to R$, given by

$$h_r(x, y_0, \dots, y_r) = h'(y_0)y_1 f^{(r)}(x) + g^{(r)}(x) +$$
$$+\sum_{i=1}^{r} h^{(r-i+1)}(y_0) \sum_{\alpha_1+\dots+\alpha_i = r-i+1} u_{\alpha_1 \dots \alpha_i,r}(x){y_1}^{\alpha_1} \dots {y_i}^{\alpha_i}$$

fulfill the generalized Hölder condition due to the variable x in I and Lipschitz condition with respect to the variable $y_i \in R, i = 0,1, \dots, r$.

3. Conclusions

In this paper the fundamental lemma connected with the form of the functions in $W_\gamma[a,b]$ has been proved. This lemma will be applied to the theorem of existence and uniqueness of solutions to functional equation: $\varphi(x) = h(\varphi[f(x)]) + g(x)$ in the forthcoming papers.

References

[1] Lupa M., A special case of generalized Hölder functions, Journal of Applied Mathematics and Computational Mechanics 2014, 13, 4, 81-89.

[2] Kuczma M., Functional Equations in a Single Variable, PWN, Warszawa 1968.

[3] Kuczma M., Choczewski B., Ger R., Iterative Functional Equations, Cambridge University Press, Cambridge-New York-Port Chester-Melbourne-Sydney 1990.

[4] Lupa M., On solutions of a functional equation in a special class of functions, Demonstratio Mathematica 1993, XXVI, 1, 137-147.

[5] Lupa M., W_γ - solutions of linear Iterative Functional Equations, Demonstratio Mathematica 1994, XXVII, 2, 417-425.