DISCRETE MODELING OF THE THREE SPECIES SYN-ECOSYSTEM WITH UNLIMITED RESOURCES

B. Hari Prasad
Department of Mathematics, Chaitanya Degree and PG College (Autonomous)
Hanamkonda, Telangana State, India-506 001
e-mail: sumathi.prasad73@yahoo.com

Abstract. In this paper, the three species syn-ecosystem is comprised of a commensal (S₁), two hosts S₂ and S₃, i.e. S₂ and S₃ both benefit S₁ without getting themselves affected either positively or adversely. Further, S₂ is a commensal of S₃, S₃ is a host of both S₁, S₂ and all the three species have unlimited resources. The basic equations for this model constitute as three first order non-linear coupled ordinary difference equations. All possible equilibrium states are identified based on the model equations at two stages and criteria for their stability are discussed. Further, the numerical solutions are computed for specific values of the various parameters and the initial conditions.

Keywords: commensal, equilibrium state, host, stable, unstable

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1. Introduction

Ecology, a branch of evolutionary biology, deals with living species that coexist in a physical environment and sustain themselves on common resources. It is a common observation that the species of the same nature can not flourish in isolation without any interaction with species of different kinds. Syn-ecology is an ecosystem comprised of two or more distinct species. Species interact with each other in one way or other. The ecological interactions can be broadly classified as ammensalism, competition, commensalism, neutralism, mutualism, predation, parasitism and so on. Lotka [1], Svirezhev and Logofet [2] and Volterra [3] pioneered theoretical ecology significantly and opened new eras in the field of life and biological sciences. The authors Rogers and Hassell [4], Varma [5] and Veilleux [6] discussed prey-predator, competing ecological models. Colinvaux [7] and Smith [8] studied basic concepts of population models in ecology.

Mathematical modeling plays a vital role in providing insight into the mutual relationships (positive, negative) between the interacting species. The general concepts of modeling have been discussed by several authors: Kapur [9], Kushing [10],

The present investigation is a discrete model of three species \( S_1, S_2, S_3 \) syn-ecosystem with unlimited resources. The system is comprised of a commensal \( S_1 \), two hosts \( S_2 \) and \( S_3 \). Further, \( S_2 \) is a commensal of \( S_3 \), \( S_3 \) is a host of both \( S_1 \) and \( S_2 \).

**Commensalism** is a symbiotic interaction between two populations where one population \( S_1 \) gets benefit from \( S_2 \), while the other \( S_2 \) is neither harmed nor benefited due to the interaction with \( S_1 \). The benefited species \( S_1 \) is called the commensal and the other \( S_2 \) is called the host. Some real-life examples of commensalism are presented below.

i. The clownfish shelters among the tentacles of the sea anemone, while the sea anemone is not affected.

ii. A squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.

iii. A flatworm attached to the horse crab and eating the crab’s food, while the crab is not put to any disadvantage.

2. Basic equations of the model

2.1. Notation adopted

\[
N_i(t) \quad - \text{the population strength of } S_i \text{ at time } t, \ i = 1, 2, 3 \\
\tau \quad - \text{time instant} \\
a_i \quad - \text{natural growth rate of } S_i, \ i = 1, 2, 3 \\
a_{12}, a_{13} \quad - \text{interaction coefficients of } S_1 \text{ due to } S_2 \text{ and } S_1 \text{ due to } S_3 \\
a_{23} \quad - \text{interaction coefficient of } S_2 \text{ due to } S_3
\]

Further, the model parameters \( a_1, a_2, a_3, a_{12}, a_{13}, a_{23} \) are assumed to be non-negative constants.

2.2. Basic equations

Consider the growth of the species during the time interval \((t, t+1)\).

**Equation for the first species** \((N_1)\):

\[
N_1(t+1) = N_1(t) + a_1 N_1(t) + a_{12} N_1(t) N_2(t) + a_{13} N_1(t) N_3(t) \quad (1)
\]
(ii) Equation for the second species ($N_2$):

$$N_2(t + 1) = N_2(t) + a_{22}N_2(t) + a_{23}N_2(t)N_3(t)$$  \hspace{1cm} (2)

(iii) Equation for the third species ($N_3$):

$$N_3(t + 1) = N_3(t) + a_3N_3(t)$$  \hspace{1cm} (3)

2.3. Species-growth equations in the discrete form

Consider the nonlinear autonomous system of discrete equations

$$N_i(t + 1) = \alpha_iN_i(t) + a_{i2}N_i(t)N_2(t) + a_{i3}N_i(t)N_3(t)$$  \hspace{1cm} (4)

$$N_2(t + 1) = \alpha_2N_2(t) + a_{23}N_2(t)N_3(t)$$  \hspace{1cm} (5)

$$N_3(t + 1) = \alpha_3N_3(t)$$  \hspace{1cm} (6)

where

$$\alpha_i = a_i + 1, \ i = 1, 2, 3$$  \hspace{1cm} (7)

3. Equilibrium states

For a continuous model the equilibrium states are defined by $\frac{dN_i}{dt} = 0$, $i = 1, 2, 3$, the equilibrium states for a discrete model are defined in terms of the period of no growth, i.e. $N_i(t + r) = N_i(t), r = 1, 2, 3, \ldots$, where $r$ is the period of the equilibrium state.

3.1. One period equilibrium states (Stage-I)

$$N_i(t + 1) = N_i(t), \ i = 1, 2, 3$$  \hspace{1cm} (8)

The system under investigation has only one equilibrium state given by

$$E_0: \vec{N}_1 = 0, \vec{N}_2 = 0, \vec{N}_3 = 0 \quad (Fully \ washed \ out \ state)$$
3.1.1. The stability of equilibrium state $E_0 (0,0,0)$

\[ N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0; \quad N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = 0; \]
\[ N_3(t) = N_3(t+1) = N_3(t+2) = \ldots = 0 \]

i.e. $N_i(t+r) = 0$, where $r$ is an integer and $i = 1, 2, 3$.
Hence, $E_0 (0,0,0)$ is stable.

3.2. Two period equilibrium states (Stage-II)

\[ N_i(t+2) = N_i(t), \quad i = 1, 2, 3 \]  \hspace{1cm} (9)

The system under investigation has five equilibrium states given by

(i) Fully washed out state
\[ E_1: \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0. \]

(ii) States in which only the second species survives
\[ E_2: \bar{N}_1 = 0, \bar{N}_2 = \frac{1 - x_2}{a_{23}}, \bar{N}_3 = 0, \text{ when } x_2 > 1 \]
\[ E_3: \bar{N}_1 = 0, \bar{N}_2 = -\left(\frac{(x_2 + 1) + \sqrt{(x_2 + 1)(x_2 - 3)}}{2a_{23}}\right), \bar{N}_3 = 0, \text{ when } x_2 > 3 \]
\[ E_4: \bar{N}_1 = 0, \bar{N}_2 = \frac{\sqrt{(x_2 + 1)(x_2 - 3)} - (x_2 + 1)}{2a_{23}}, \bar{N}_3 = 0, \text{ when } x_2 > 3 \]
\[ E_5: \bar{N}_1 = 0, \bar{N}_2 = -\frac{2}{a_{23}}, \bar{N}_3 = 0, \text{ when } x_2 = 3 \]

The states $E_1$ and $E_2$ coincide when $x_2 = 3$ and do not exist when $x_2 < 3$.

3.2.1. The stability of equilibrium states

The equilibrium state $E_1$ is stable. Now we will discuss the stability of all other equilibrium states.

The stability of $E_2$:

\[ N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0; \quad N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = 0 \]
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\[ N_i(t + r) = 0, \text{ where } r \text{ is an integer and } i = 1,3 \]

\[ N_2(t + 1) = \frac{\alpha_2 (1 - \alpha_2)}{a_{23}}; \quad N_2(t + 2) = \frac{\alpha_2^2 (1 - \alpha_2)}{a_{23}}; \quad N_2(t + 3) = \frac{\alpha_2^3 (1 - \alpha_2)}{a_{23}} \]

and so on

\[ N_2(t + r) = \frac{\alpha_2^r (1 - \alpha_2)}{a_{23}}, \text{ where } r \text{ is an integer.} \]

Hence, \( E_2 \) is **unstable**.

**The stability of \( E_3 \):**

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0; \quad N_3(t) = N_3(t + 1) = N_3(t + 2) = \ldots = 0 \]

\[ N_2(t + 1) = -\alpha_2 \left[ \frac{(\alpha_2 + 1) + \sqrt{(\alpha_2 + 1)(\alpha_2 - 3)}}{2a_{23}} \right]; \]

\[ N_2(t + 2) = -\alpha_2^2 \left[ \frac{(\alpha_2 + 1) + \sqrt{(\alpha_2 + 1)(\alpha_2 - 3)}}{2a_{23}} \right]; \]

\[ N_2(t + 3) = -\alpha_2^3 \left[ \frac{(\alpha_2 + 1) + \sqrt{(\alpha_2 + 1)(\alpha_2 - 3)}}{2a_{23}} \right] \text{ and so on} \]

\[ N_2(t + r) = -\alpha_2^r \left[ \frac{(\alpha_2 + 1) + \sqrt{(\alpha_2 + 1)(\alpha_2 - 3)}}{2a_{23}} \right], \text{ where } r \text{ is an integer.} \]

Hence, \( E_3 \) is **unstable**.

**The stability of \( E_4 \):**

\[ N_1(t) = N_1(t + 1) = N_1(t + 2) = \ldots = 0; \quad N_4(t) = N_4(t + 1) = N_4(t + 2) = \ldots = 0 \]

\[ N_2(t + 1) = \alpha_2 \left[ \frac{\sqrt{(\alpha_2 + 1)(\alpha_2 - 3)} - (\alpha_2 + 1)}{2a_{23}} \right]; \]

\[ N_2(t + 2) = \alpha_2^2 \left[ \frac{\sqrt{(\alpha_2 + 1)(\alpha_2 - 3)} - (\alpha_2 + 1)}{2a_{23}} \right]; \]

\[ N_2(t + 3) = \alpha_2^3 \left[ \frac{\sqrt{(\alpha_2 + 1)(\alpha_2 - 3)} - (\alpha_2 + 1)}{2a_{23}} \right] \text{ and so on} \]
i.e. \( N_{t+r} = \alpha^r \left[ \frac{\sqrt{(\alpha_2 + 1)(\alpha_2 - 3) - (\alpha_2 + 1)}}{2\alpha_2} \right] \), where \( r \) is an integer.

Hence, \( E_4 \) is **unstable**.

**The stability of \( E_5 \):**

\[
N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0; \quad N_2(t) = N_2(t+1) = N_2(t+2) = ... = 0
\]

i.e. \( N_i(t+r) = 0 \), where \( r \) is an integer and \( i = 1,3 \)

\[
N_2(t+1) = -\frac{2\alpha_2}{\alpha_3}; \quad N_2(t+2) = -\frac{2\alpha_2^2}{\alpha_3}; \quad N_2(t+3) = -\frac{2\alpha_2^3}{\alpha_3} \quad \text{and so on}
\]

i.e. \( N_2(t+r) = -\frac{2\alpha_2^r}{\alpha_3} \), where \( r \) is an integer.

Hence, \( E_5 \) is **unstable**.

At this stage, in all five equilibrium states, the fully washed out state \( E_1 \) is stable only and all the remaining are unstable.

### 4. Computer simulations

The numerical simulations of the discrete model equations computed for specific values of the various parameters and the initial conditions. The results are illustrated in Figures 1 to 5.

![Variation of N1, N2, N3 against time](image)
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Fig. 2. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $\alpha_1 = 5.9$, $\alpha_2 = 2$, $\alpha_3 = 2.6$, $a_{12} = 1.5$, $a_{13} = 3.8$, $a_{23} = 1$, $N_1(0) = 0$, $N_2(0) = -1$, $N_3(0) = 0$

Fig. 3. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $\alpha_1 = 3.9$, $\alpha_2 = 3.1$, $\alpha_3 = 5$, $a_{12} = 1.9$, $a_{13} = 0.2$, $a_{23} = 40$, $N_1(0) = 0$, $N_2(0) = -0.05$, $N_3(0) = 0$

Fig. 4. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $\alpha_1 = 0.6$, $\alpha_2 = 3$, $\alpha_3 = 5.9$, $a_{12} = 2$, $a_{13} = 8.5$, $a_{23} = 48$, $N_1(0) = 0$, $N_2(0) = -0.04$, $N_3(0) = 0$
Fig. 5. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $\alpha_1 = 9.3$, $\alpha_2 = 1.3$, $\alpha_3 = 7.9$, $a_{12} = 2.8$, $a_{13} = 3.7$, $a_{23} = 3$, $N_1(0) = 0$, $N_2(0) = 0$, $N_3(0) = 0$

5. Conclusion

The present paper deals with an investigation on a discrete model of a three-species syn-ecosystem with unlimited resources. The system is comprised of a commensal ($S_1$) common to two hosts $S_2$ and $S_3$, i.e. $S_2$ and $S_3$ both benefit $S_1$ without getting themselves affected either positively or adversely. All possible equilibrium points of the model are identified based on the model equations at two stages.

Stage-I : $N_i(t+1) = N_i(t); \ i = 1, 2, 3$

Stage-II : $N_i(t+2) = N_i(t); \ i = 1, 2, 3$

In Stage-I there is only one equilibrium point, while in Stage-II there would be five equilibrium points. The equilibrium point $E_0$ in Stage-I is found to be stable while in Stage-II only one is stable. Further the numerical solutions for the discrete model equations are computed.

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References

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