Journal of Applied Mathematics and Computational Mechanics 2015, 14(2), 85-93
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2015.2.09 e-ISSN 2353-0588

DISCRETE MODELING OF THE THREE SPECIES SYN-ECOSYSTEM WITH UNLIMITED RESOURCES

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Abstract. In this paper, the three species syn-ecosystem is comprised of a commensal (S_1), two hosts S_2 and S_3, i.e. S_2 and S_3 both benefit S_1 without getting themselves affected either positively or adversely. Further, S_2 is a commensal of S_3, S_3 is a host of both S_1, S_2 and all the three species have unlimited resources. The basic equations for this model constitute as three first order non-linear coupled ordinary difference equations. All possible equilibrium states are identified based on the model equations at two stages and criteria for their stability are discussed. Further, the numerical solutions are computed for specific values of the various parameters and the initial conditions.

Keywords: *commensal, equilibrium state, host, stable, unstable*

2010 Mathematics Subject Classification: 92D25, 92D40

1. Introduction

Ecology, a branch of evolutionary biology, deals with living species that coexist in a physical environment and sustain themselves on common resources. It is a common observation that the species of the same nature can not flourish in isolation without any interaction with species of different kinds. Syn-ecology is an ecosystem comprised of two or more distinct species. Species interact with each other in one way or other. The ecological interactions can be broadly classified as ammensalism, competition, commensalism, neutralism, mutualism, predation, parasitism and so on. Lotka [1], Svirezhev and Logofet [2] and Volterra [3] pioneered theoretical ecology significantly and opened new eras in the field of life and biological sciences. The authors Rogers and Hassell [4], Varma [5] and Veilleux [6] discussed prey-predator, competing ecological models. Colinvaux [7] and Smith [8] studied basic concepts of population models in ecology.

Mathematical modeling plays a vital role in providing insight into the mutual relationships (positive, negative) between the interacting species. The general concepts of modeling have been discussed by several authors: Kapur [9], Kushing [10], Meyer [11] and Pielou [12]. Srinivas [13] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan and Pattabhiramacharyulu [14] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Acharyulu and Pattabhiramacharyulu [15, 16] derived some productive results on various mathematical models of ecological ammensalism with multifarious resources in the manifold directions. Further, Kumar [17] studied some mathematical models of ecological commensalism. The present author Prasad [18-21] investigated continuous and discrete models on the three species syn-ecosystems.

The present investigation is a discrete model of three species (S_1, S_2, S_3) syn-ecosystem with unlimited resources. The system is comprised of a commensal (S_1), two hosts S_2 and S_3. Further, S_2 is a commensal of S_3, S_3 is a host of both S_1 and S_2.

Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2), while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other (S_2) is called the host. Some real-life examples of commensalism are presented below.

i. The clownfish shelters among the tentacles of the sea anemone, while the sea anemone is not affected.
ii. A squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.
iii. A flatworm attached to the horse crab and eating the crab's food, while the crab is not put to any disadvantage.

2. Basic equations of the model

2.1. Notation adopted

$N_i(t)$ - the population strength of S_i at time t, $i = 1,2,3$
t - time instant
a_i - natural growth rate of S_i, $i = 1,2,3$
a_{12}, a_{13} - interaction coefficients of S_1 due to S_2 and S_1 due to S_3
a_{23} - interaction coefficient of S_2 due to S_3

Further, the model parameters $a_1, a_2, a_3, a_{12}, a_{13}, a_{23}$ are assumed to be non-negative constants.

2.2. Basic equations

Consider the growth of the species during the time interval $(t, t+1)$.

(i) Equation for the first species (N_1):

$$N_1(t+1) = N_1(t) + a_1 N_1(t) + a_{12} N_1(t)\, N_2(t) + a_{13} N_1(t)\, N_3(t) \tag{1}$$

(ii) Equation for the second species (N_2):

$$N_2(t+1) = N_2(t) + a_2 N_2(t) + a_{23} N_2(t) N_3(t) \tag{2}$$

(iii) Equation for the third species (N_3):

$$N_3(t+1) = N_3(t) + a_3 N_3(t) \tag{3}$$

2.3. Species-growth equations in the discrete form

Consider the nonlinear autonomous system of discrete equations

$$N_1(t+1) = \alpha_1 N_1(t) + a_{12} N_1(t)\, N_2(t) + a_{13} N_1(t)\, N_3(t) \tag{4}$$

$$N_2(t+1) = \alpha_2 N_2(t) + a_{23} N_2(t) N_3(t) \tag{5}$$

$$N_3(t+1) = \alpha_3 N_3(t) \tag{6}$$

where

$$\alpha_i = a_i + 1, i = 1,2,3 \tag{7}$$

3. Equilibrium states

For a continuous model the equilibrium states are defined by $\dfrac{dN_i}{dt} = 0,\ i = 1,2,3$, the equilibrium states for a discrete model are defined in terms of the period of no growth, i.e. $N_i(t+r) = N_i(t), r = 1,2,3,...$, where r is the period of the equilibrium state.

3.1. One period equilibrium states (Stage-I)

$$N_i(t+1) = N_i(t),\ i = 1,2,3 \tag{8}$$

The system under investigation has **only one** equilibrium state given by

$$E_0 : \overline{N}_1 = 0,\ \overline{N}_2 = 0,\ \overline{N}_3 = 0 \quad (\textit{Fully washed out state})$$

3.1.1. The stability of equilibrium state $E_0\left(0,0,0\right)$

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_2(t)=N_2(t+1)=N_2(t+2)=...=0;$$
$$N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

i.e. $N_i(t+r)=0$, where r is an integer and $i=1,2,3$.
Hence, $E_0\left(0,0,0\right)$ is **stable**.

3.2. Two period equilibrium states (Stage-II)

$$N_i(t+2)=N_i(t),\ i=1,2,3 \tag{9}$$

The system under investigation has **five** equilibrium states given by

(i) Fully washed out state

$$E_1:\overline{N}_1=0,\ \overline{N}_2=0,\ \overline{N}_3=0\,.$$

(ii) States in which only the second species survives

$$E_2:\overline{N}_1=0,\ \overline{N}_2=\frac{1-\alpha_2}{a_{23}},\ \overline{N}_3=0\,,\ \text{when } \alpha_2>1$$

$$E_3:\overline{N}_1=0,\overline{N}_2=-\left[\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{23}}\right],\overline{N}_3=0\,,\ \text{when } \alpha_2>3$$

$$E_4:\overline{N}_1=0,\overline{N}_2=\frac{\sqrt{(\alpha_2+1)(\alpha_2-3)}-(\alpha_2+1)}{2a_{23}},\overline{N}_3=0\,,\ \text{when } \alpha_2>3$$

$$E_5:\overline{N}_1=0,\overline{N}_2=-\frac{2}{a_{23}},\ \overline{N}_3=0\,,\ \text{when } \alpha_2=3$$

The states E_3 and E_4 coincide when $\alpha_2=3$ and do not exist when $\alpha_2<3$.

3.2.1. The stability of equilibrium states

The equilibrium state E_1 is **stable**. Now we will discuss the stability of all other equilibrium states.

The stability of E_2:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

i.e. $N_i(t+r)=0$, where r is an integer and $i=1,3$

$$N_2(t+1)=\frac{\alpha_2\left(1-\alpha_2\right)}{a_{23}};\ N_2(t+2)=\frac{\alpha_2^2\left(1-\alpha_2\right)}{a_{23}};\ N_2(t+3)=\frac{\alpha_2^3\left(1-\alpha_2\right)}{a_{23}}\ \text{and so on}$$

i.e. $N_2(t+r)=\dfrac{\alpha_2^r(1-\alpha_2)}{a_{23}}$, where r is an integer.

Hence, E_2 is **unstable**.

The stability of E_3:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

i.e. $N_i(t+r)=0$, where r is an integer and $i=1,3$

$$N_2(t+1)=-\alpha_2\left[\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{23}}\right];$$

$$N_2(t+2)=-\alpha_2^2\left[\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{23}}\right];$$

$$N_2(t+3)=-\alpha_2^3\left[\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{23}}\right]\ \text{and so on}$$

i.e. $N_2(t+r)=-\alpha_2^r\left[\dfrac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{23}}\right]$, where r is an integer.

Hence, E_3 is **unstable**.

The stability of E_4:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

i.e. $N_i(t+r)=0$, where r is an integer and $i=1,3$

$$N_2(t+1)=\alpha_2\left[\frac{\sqrt{(\alpha_2+1)(\alpha_2-3)}-(\alpha_2+1)}{2a_{23}}\right];$$

$$N_2(t+2)=\alpha_2^2\left[\frac{\sqrt{(\alpha_2+1)(\alpha_2-3)}-(\alpha_2+1)}{2a_{23}}\right];$$

$$N_2(t+3)=\alpha_2^3\left[\frac{\sqrt{(\alpha_2+1)(\alpha_2-3)}-(\alpha_2+1)}{2a_{23}}\right]\ \text{and so on}$$

i.e. $N_2(t+r) = \alpha_2^r \left[\frac{\sqrt{(\alpha_2+1)(\alpha_2-3)} - (\alpha_2+1)}{2a_{23}} \right]$, where r is an integer.

Hence, E_4 is **unstable**.

The stability of E_5:

$$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0;\ N_3(t) = N_3(t+1) = N_3(t+2) = ... = 0$$

i.e. $N_i(t+r) = 0$, where r is an integer and $i = 1,3$

$N_2(t+1) = -\frac{2\alpha_2}{a_{23}};\ N_2(t+2) = -\frac{2\alpha_2^2}{a_{23}};\ N_2(t+3) = -\frac{2\alpha_2^3}{a_{23}}$ and so on

i.e. $N_2(t+r) = -\frac{2\alpha_2^r}{a_{23}}$, where r is an integer.

Hence, E_5 is **unstable**.

At this stage, in all five equilibrium states, the fully washed out state E_1 is **stable** only and all the remaining are **unstable**.

4. Computer simulations

The numerical simulations of the discrete model equations computed for specific values of the various parameters and the initial conditions. The results are illustrated in Figures 1 to 5.

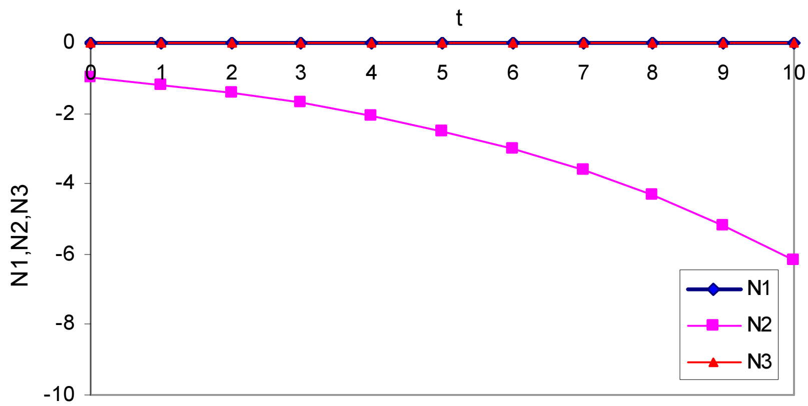

Fig. 1. Variation of N_1, N_2 and N_3 against time (t) for $\alpha_1 = 1.2$, $\alpha_2 = 6.9$, $\alpha_3 = 3.6$, $a_{12} = 4.2$, $a_{13} = 5.8$, $a_{23} = 0.2$, $N_1(0) = 0$, $N_2(0) = -1$, $N_3(0) = 0$

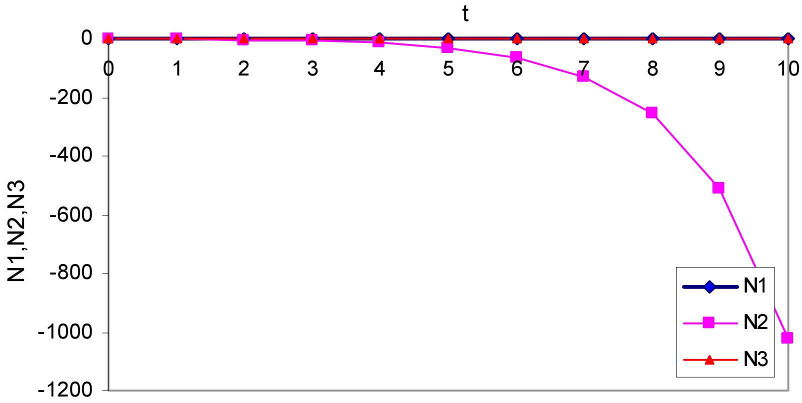

Fig. 2. Variation of N_1, N_2 and N_3 against time (t) for $\alpha_1 = 5.9$, $\alpha_2 = 2$, $\alpha_3 = 2.6$, $a_{12} = 1.5$, $a_{13} = 3.8$, $a_{23} = 1$, $N_1(0) = 0$, $N_2(0) = -1$, $N_3(0) = 0$

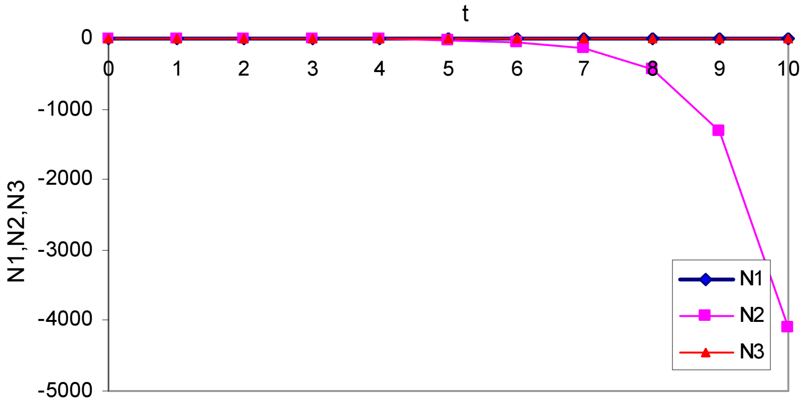

Fig. 3. Variation of N_1, N_2 and N_3 against time (t) for $\alpha_1 = 3.9$, $\alpha_2 = 3.1$, $\alpha_3 = 5$, $a_{12} = 1.9$, $a_{13} = 0.2$, $a_{23} = 40$, $N_1(0) = 0$, $N_2(0) = -0.05$, $N_3(0) = 0$

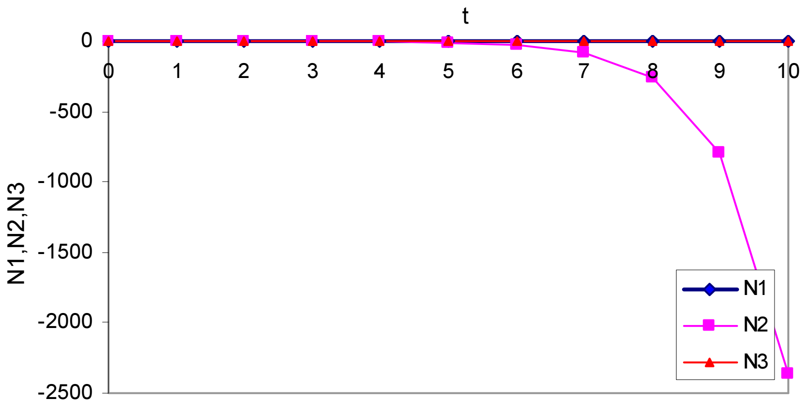

Fig. 4. Variation of N_1, N_2 and N_3 against time (t) for $\alpha_1 = 0.6$, $\alpha_2 = 3$, $\alpha_3 = 5.9$, $a_{12} = 2$, $a_{13} = 8.5$, $a_{23} = 48$, $N_1(0) = 0$, $N_2(0) = -0.04$, $N_3(0) = 0$

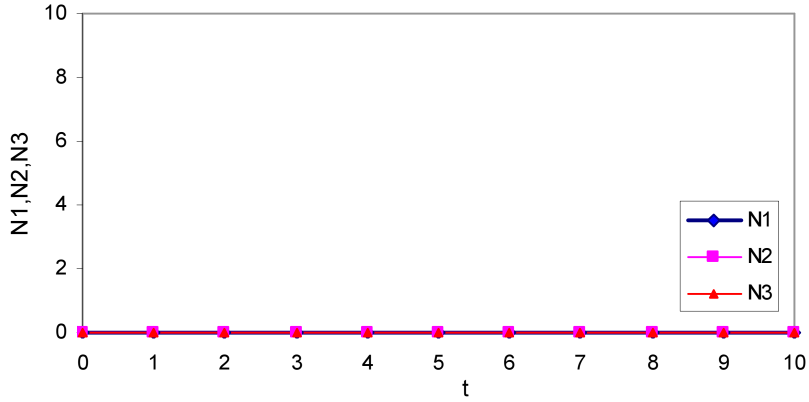

Fig. 5. Variation of N_1, N_2 and N_3 against time (t) for $\alpha_1 = 9.3$, $\alpha_2 = 1.3$, $\alpha_3 = 7.9$, $a_{12} = 2.8$, $a_{13} = 3.7$, $a_{23} = 3$, $N_1(0) = 0$, $N_2(0) = 0$, $N_3(0) = 0$

5. Conclusion

The present paper deals with an investigation on a discrete model of a three species syn-ecosystem with unlimited resources. The system is comprised of a commensal (S_1) common to two hosts S_2 and S_3, i.e. S_2 and S_3 both benefit S_1 without getting themselves affected either positively or adversely. All possible equilibrium points of the model are identified based on the model equations at two stages.

$$\text{Stage-I}: \; N_i(t+1) = N_i(t); \; i = 1,2,3$$

$$\text{Stage-II}: \; N_i(t+2) = N_i(t); \; i = 1,2,3$$

In Stage-I there is only one equilibrium point, while in Stage-II there would be five equilibrium points. The equilibrium point E_0 in Stage-I is found to be **stable** while in Stage-II only one is **stable**. Further the numerical solutions for the discrete model equations are computed.

Acknowledgment

I thank to Professor (Retd) N.Ch. Pattabhi Ramacharyulu, Department of Mathematics, National Institute of Technology, Warangal (T.S.), India for his valuable suggestions and encouragement.

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